Interactive Formal Verification
7: Inductive Definitions

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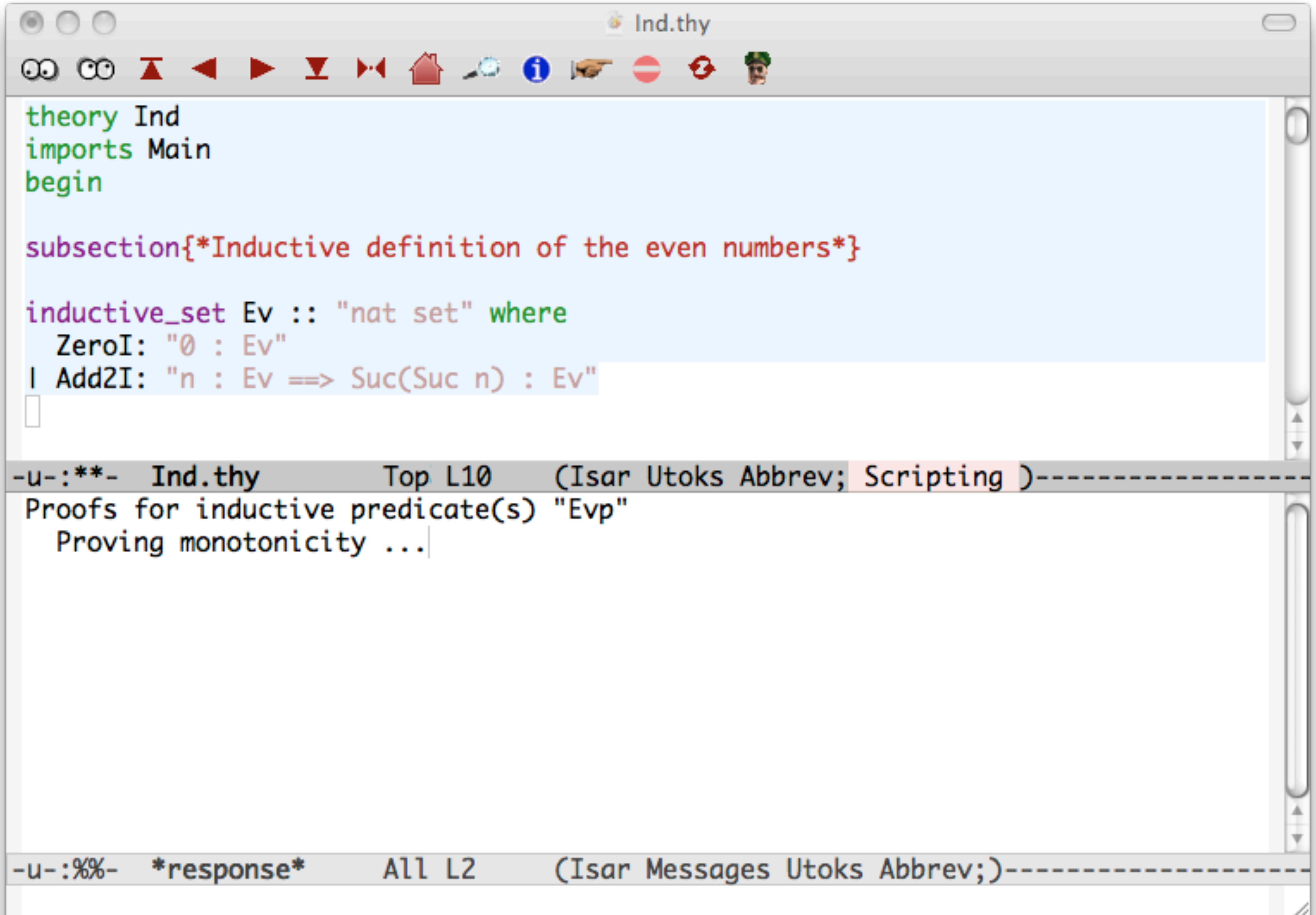
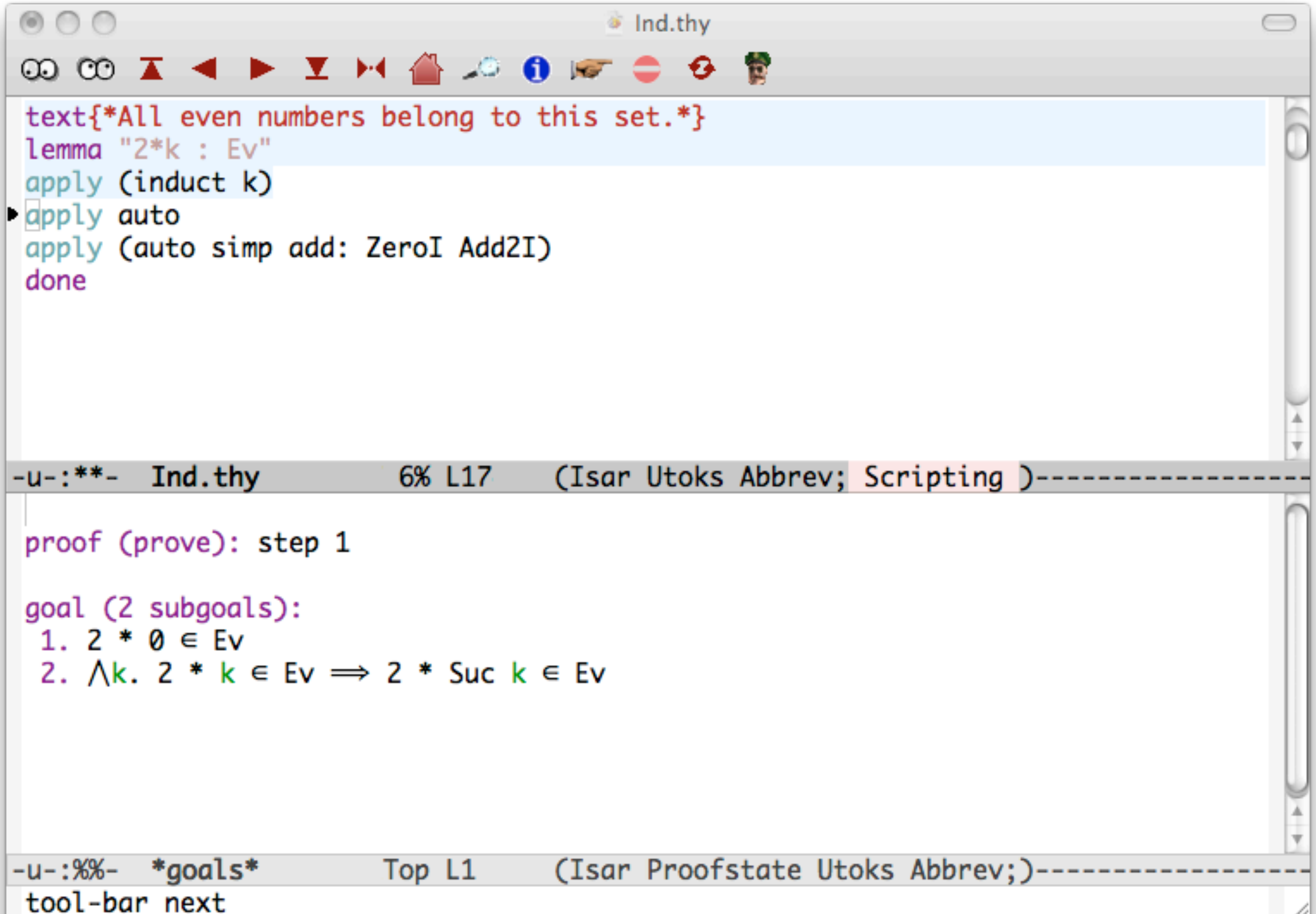
- These can be viewed as *introduction rules*.
- We get an *induction principle* to express that no other numbers are even.
- Induction is used throughout mathematics, and to express the semantics of programming languages.

Inductive Definitions in Isabelle



Even Numbers Belong to `Ev`



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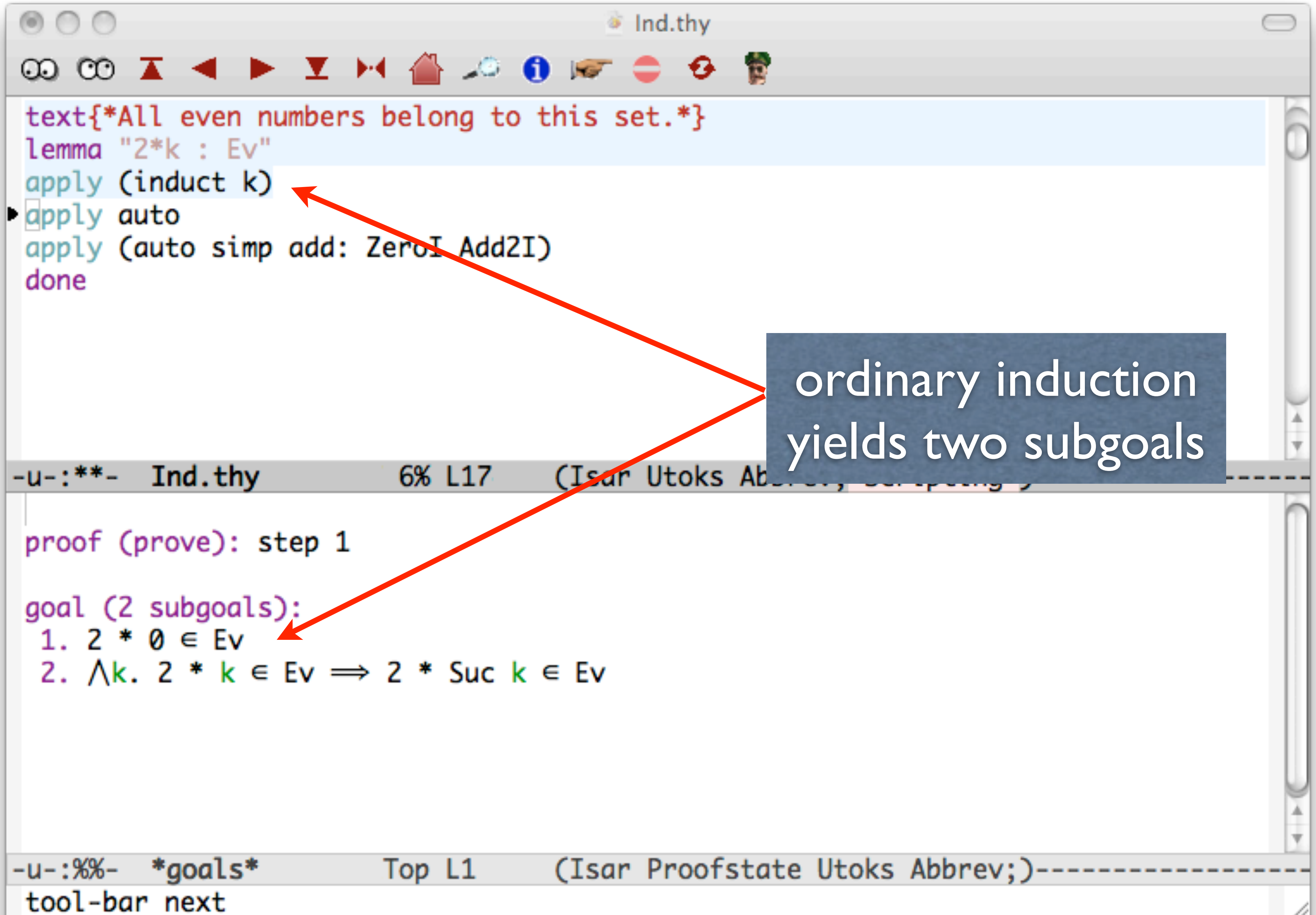
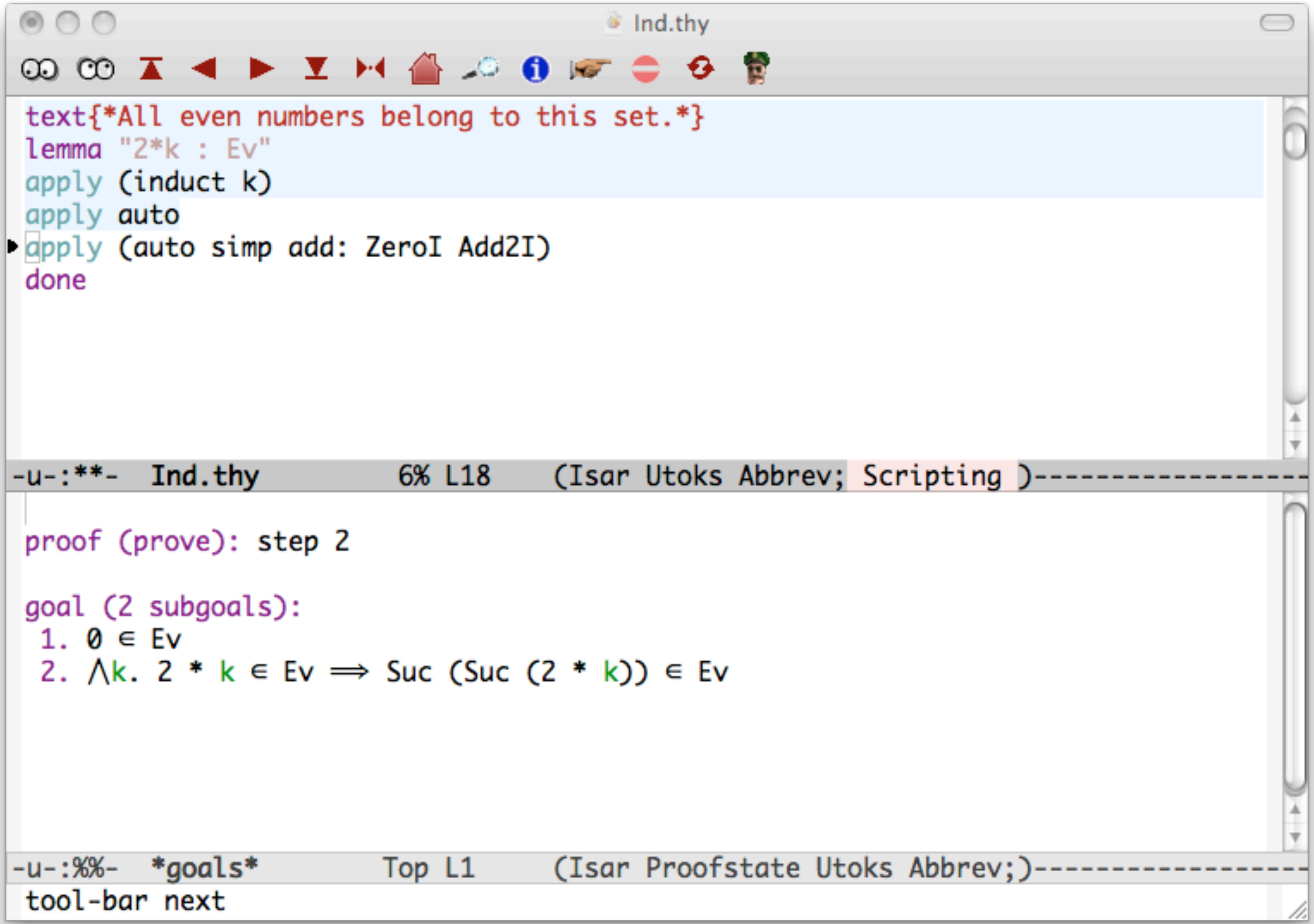


Proving Set Membership



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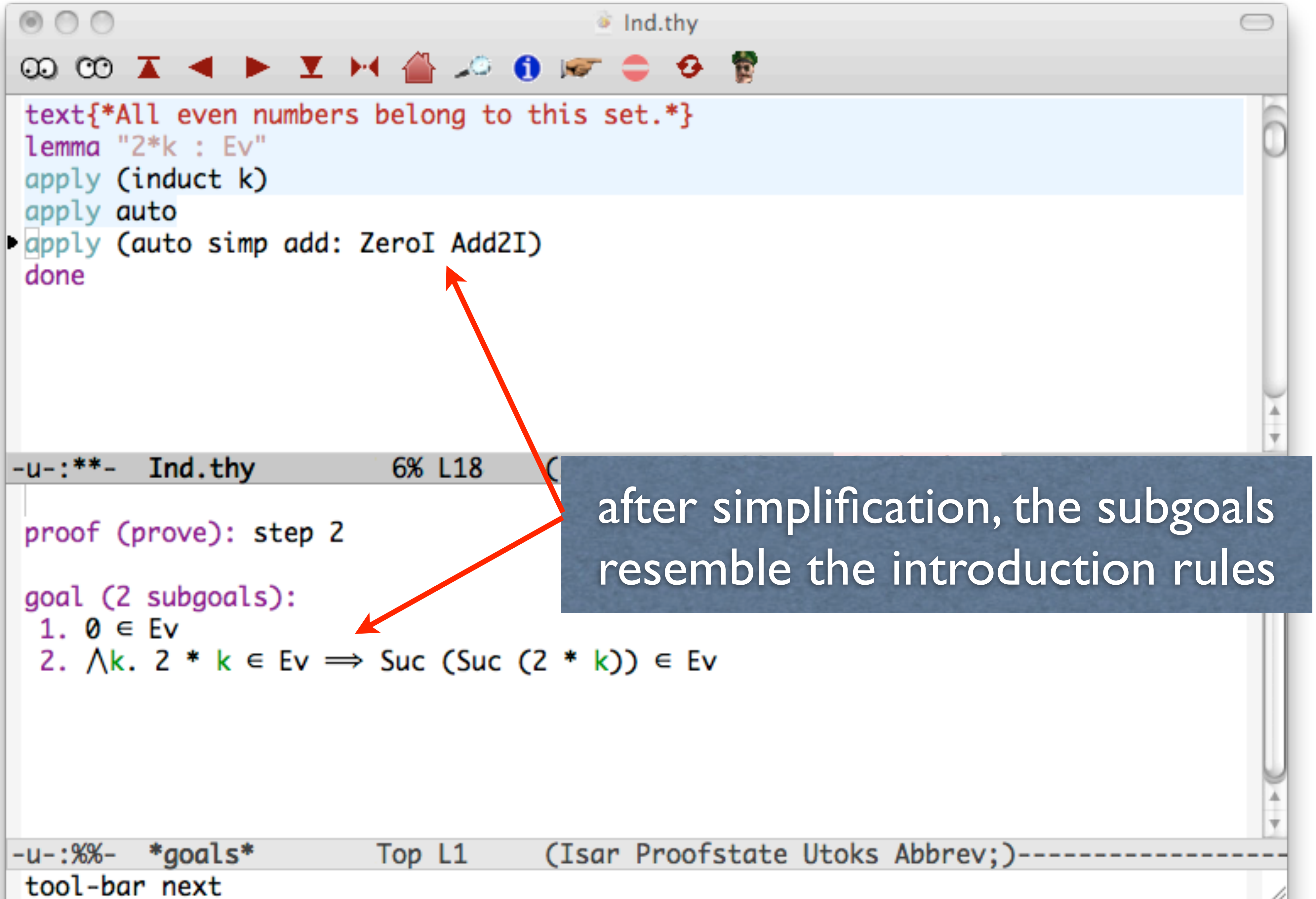
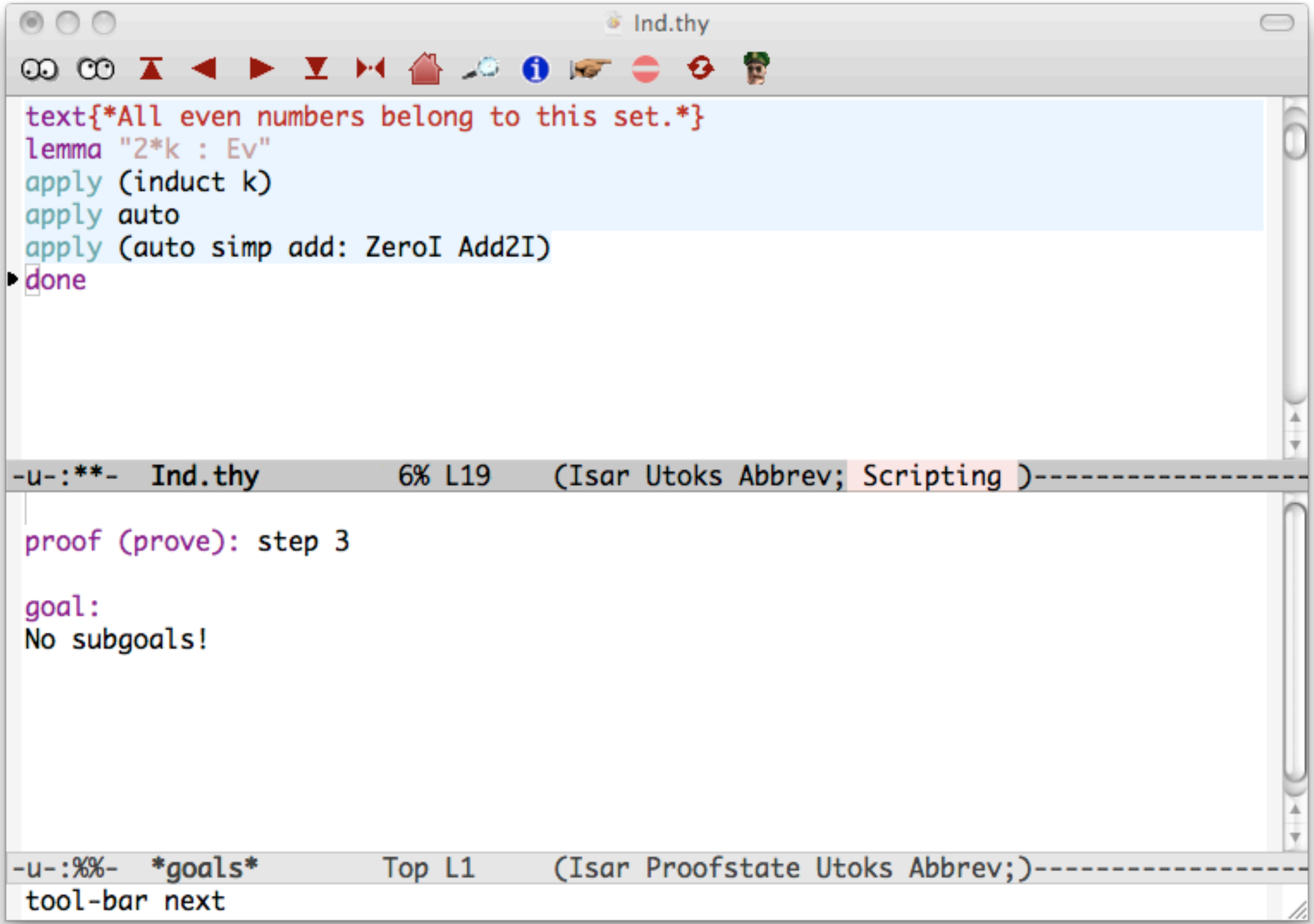


Finishing the Proof



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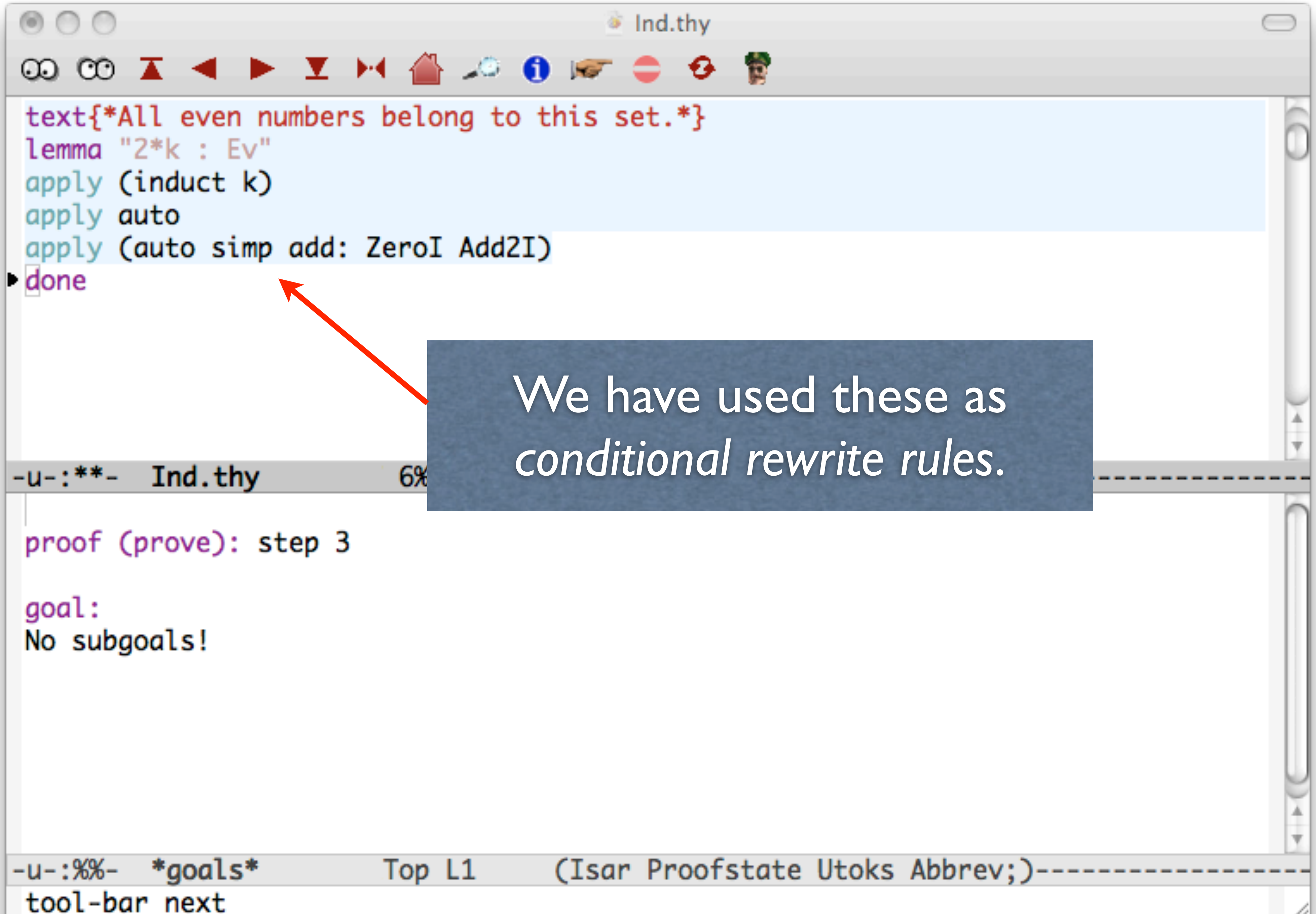
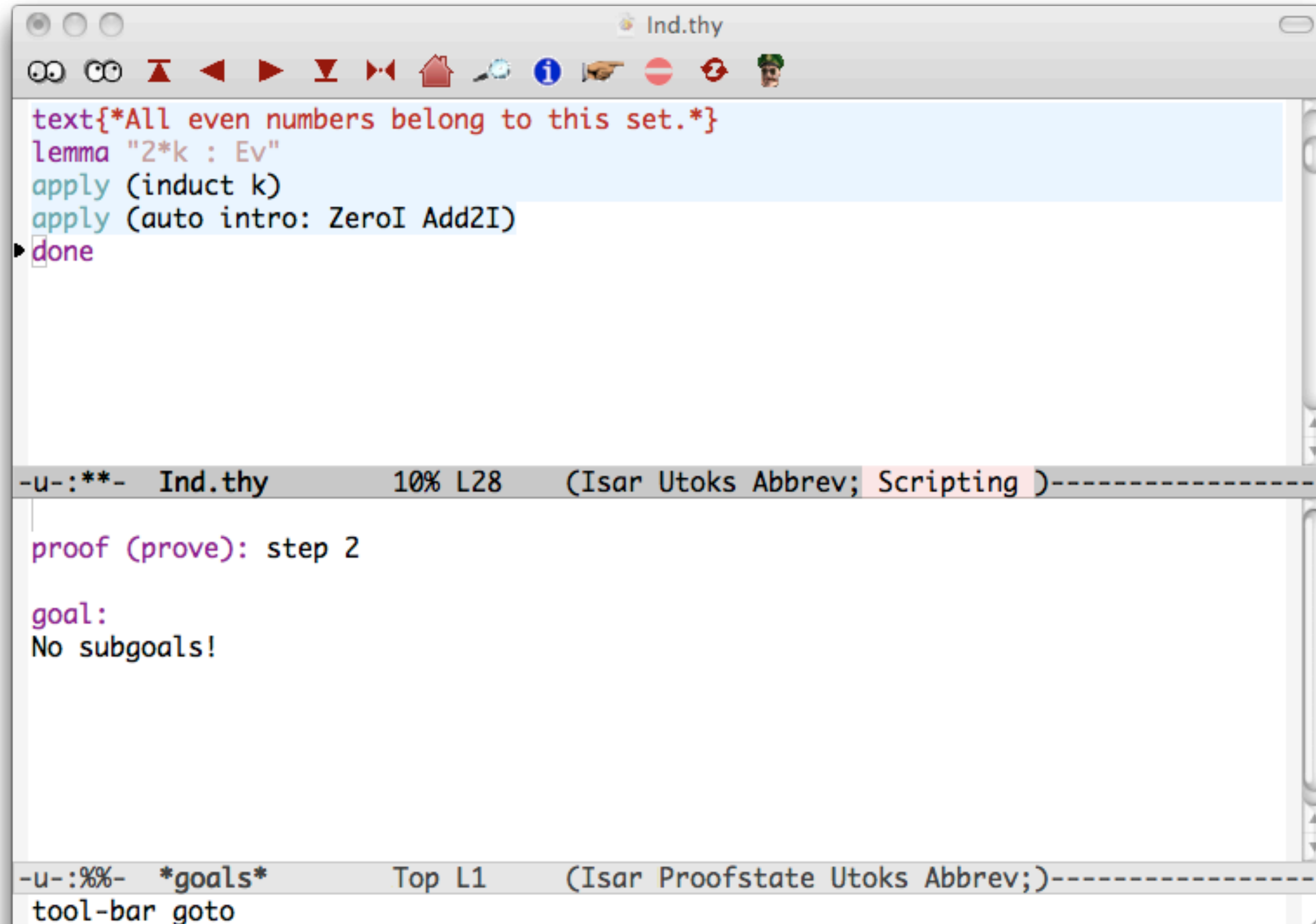


We have used these as *conditional rewrite rules.*

Finishing the Proof



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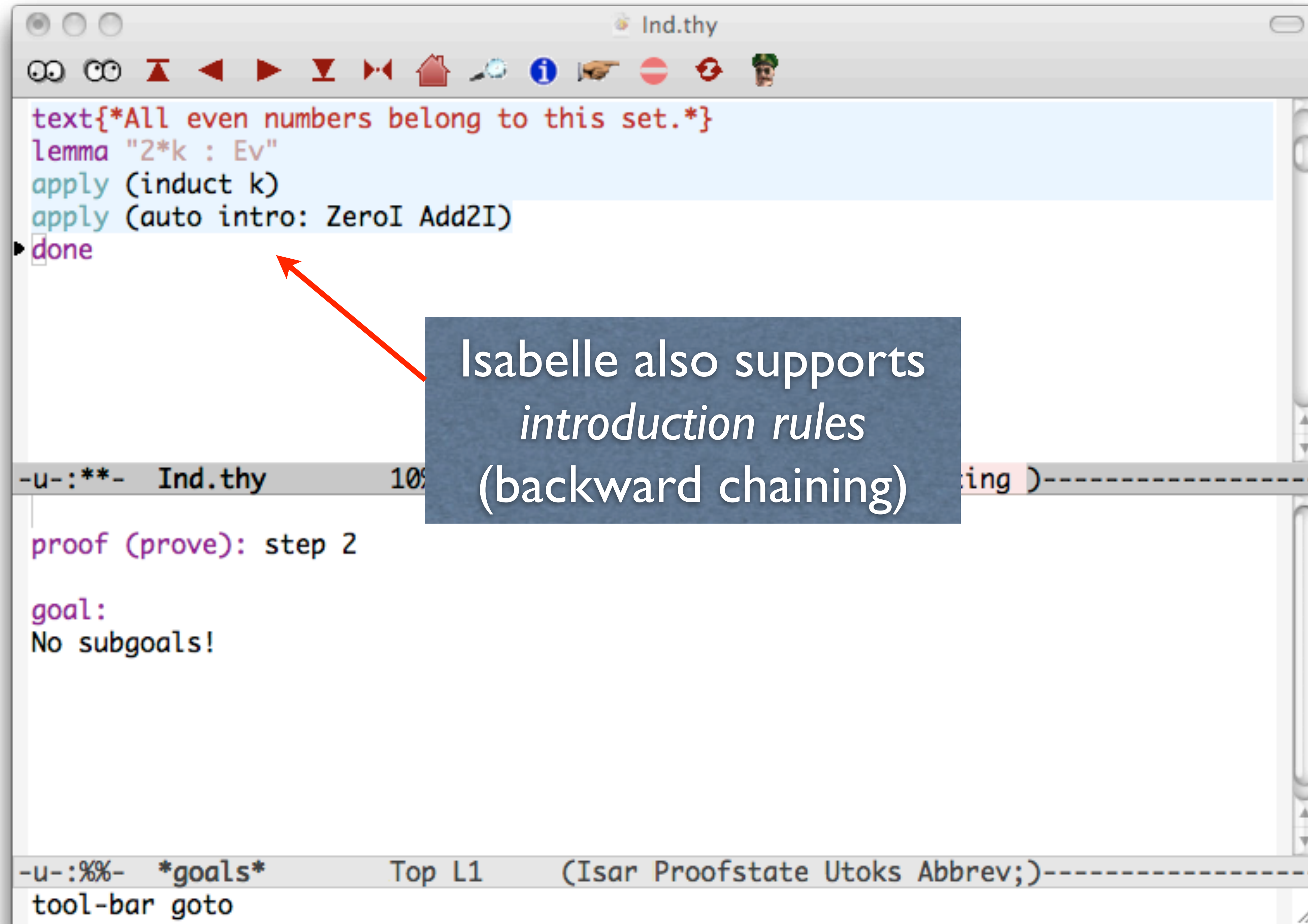

Rule Induction

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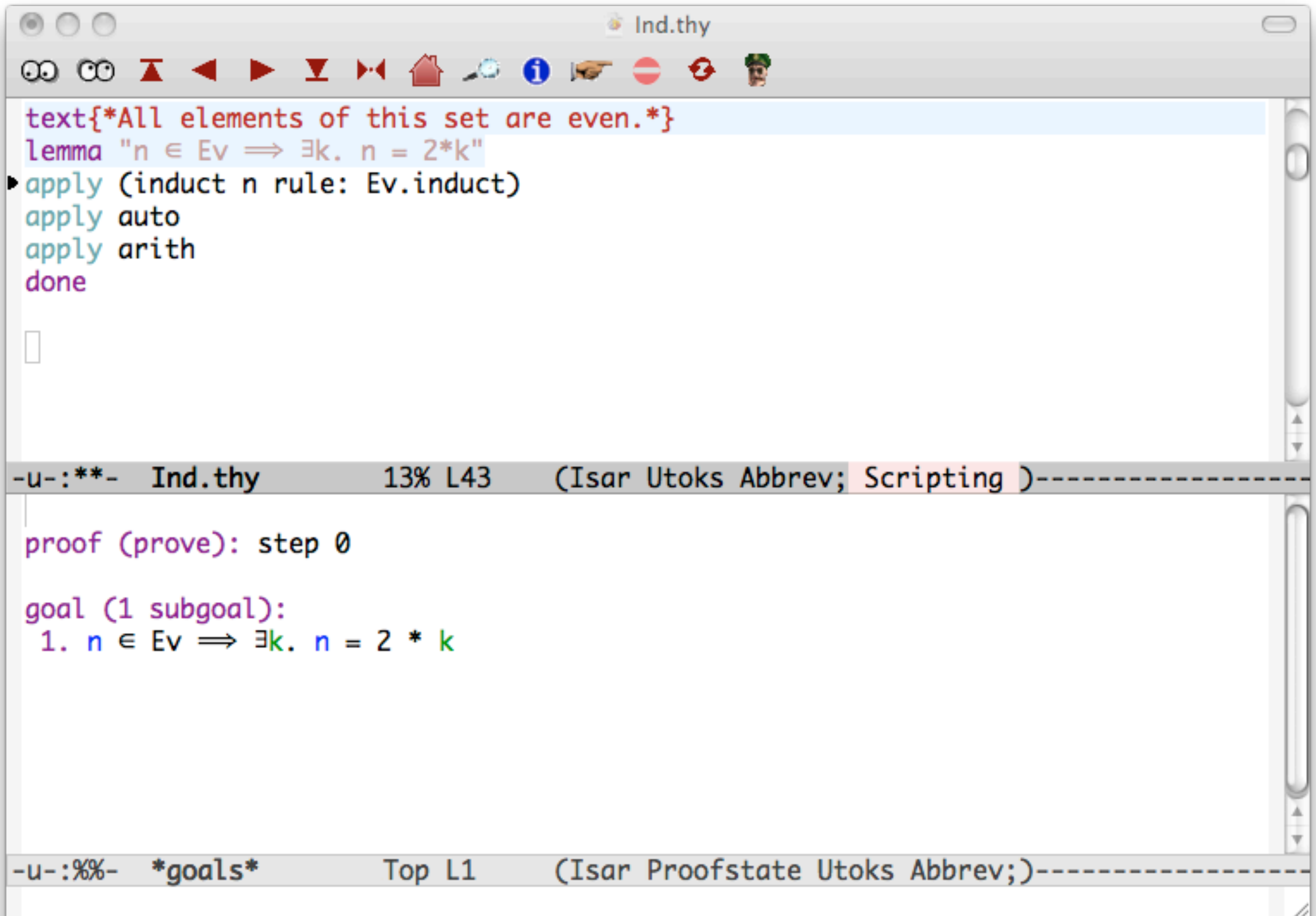
- It expresses that the inductive set is *minimal*.
- It is sometimes called "induction on derivations"
- There is a *base case* for every non-recursive introduction rule
- ...and an *inductive step* for the other rules.

Ev Has only Even Numbers



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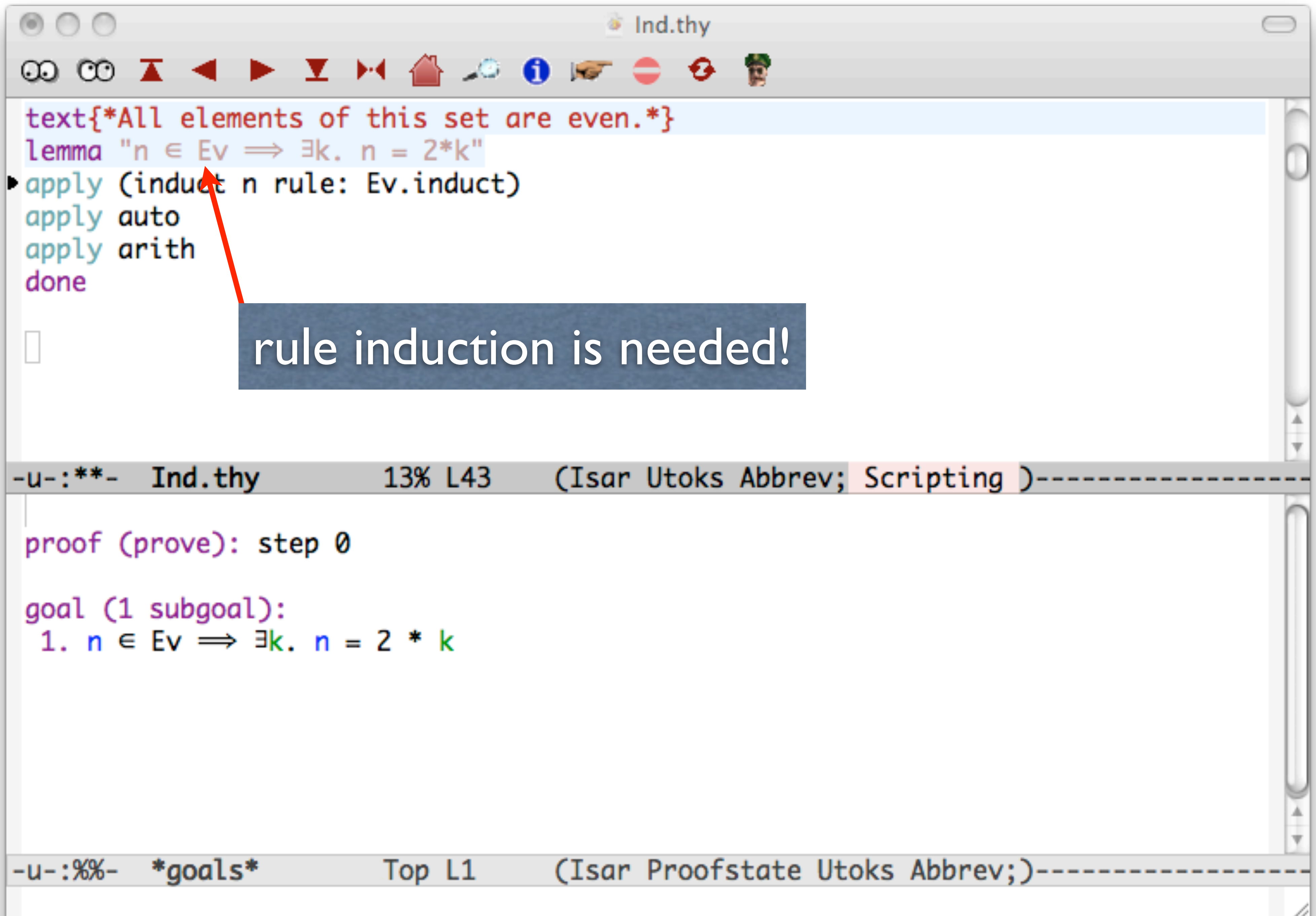

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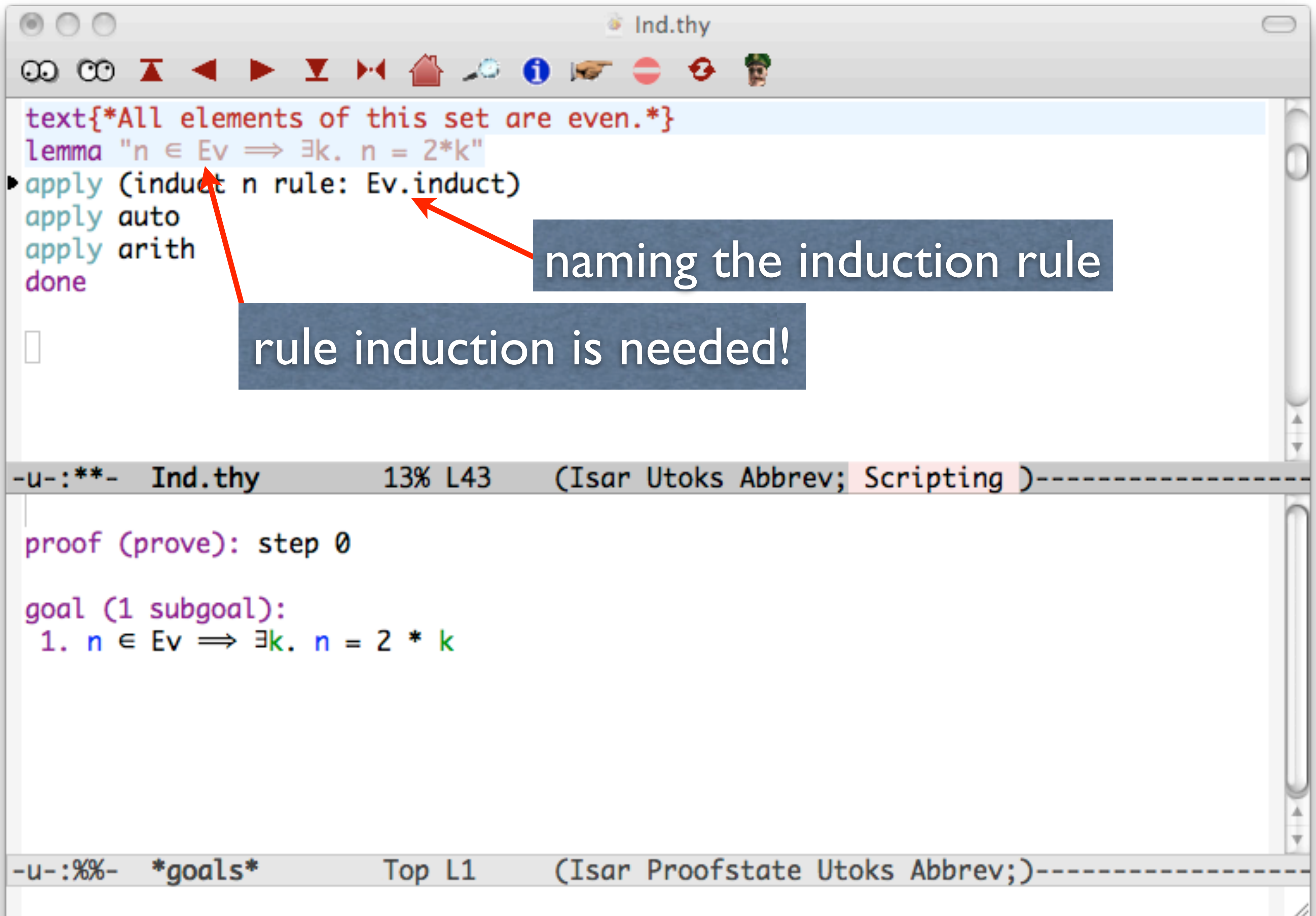

An Example of Rule Induction

```
Ind.thy

text{*All elements of this set are even.*}
lemma "n ∈ Ev ⟹ ∃k. n = 2*k"
apply (induct n rule: Ev.induct)
apply auto
apply arith
done

-u-:**-  Ind.thy        13% L39      (Isar Utoks Abbrev; Scripting )----------------

proof (prove): step 1

goal (2 subgoals):
 1. ∃k. 0 = 2 * k
 2. ⋀n. ⟦n ∈ Ev; ∃k. n = 2 * k⟧ ⟹ ∃k. Suc (Suc n) = 2 * k

-u-:%%-  *goals*        Top L1       (Isar Proofstate Utoks Abbrev;)----------------
tool-bar next
```

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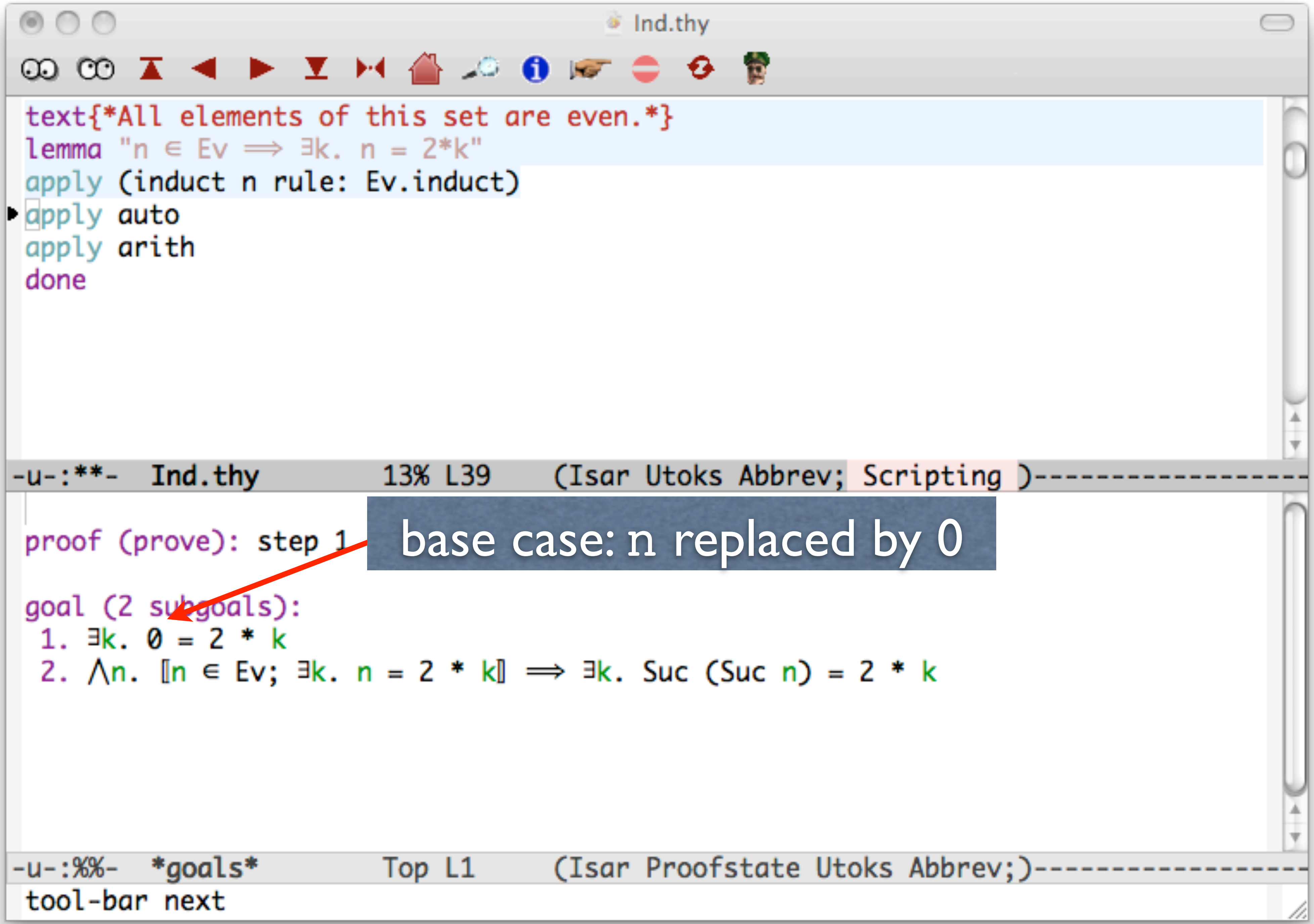

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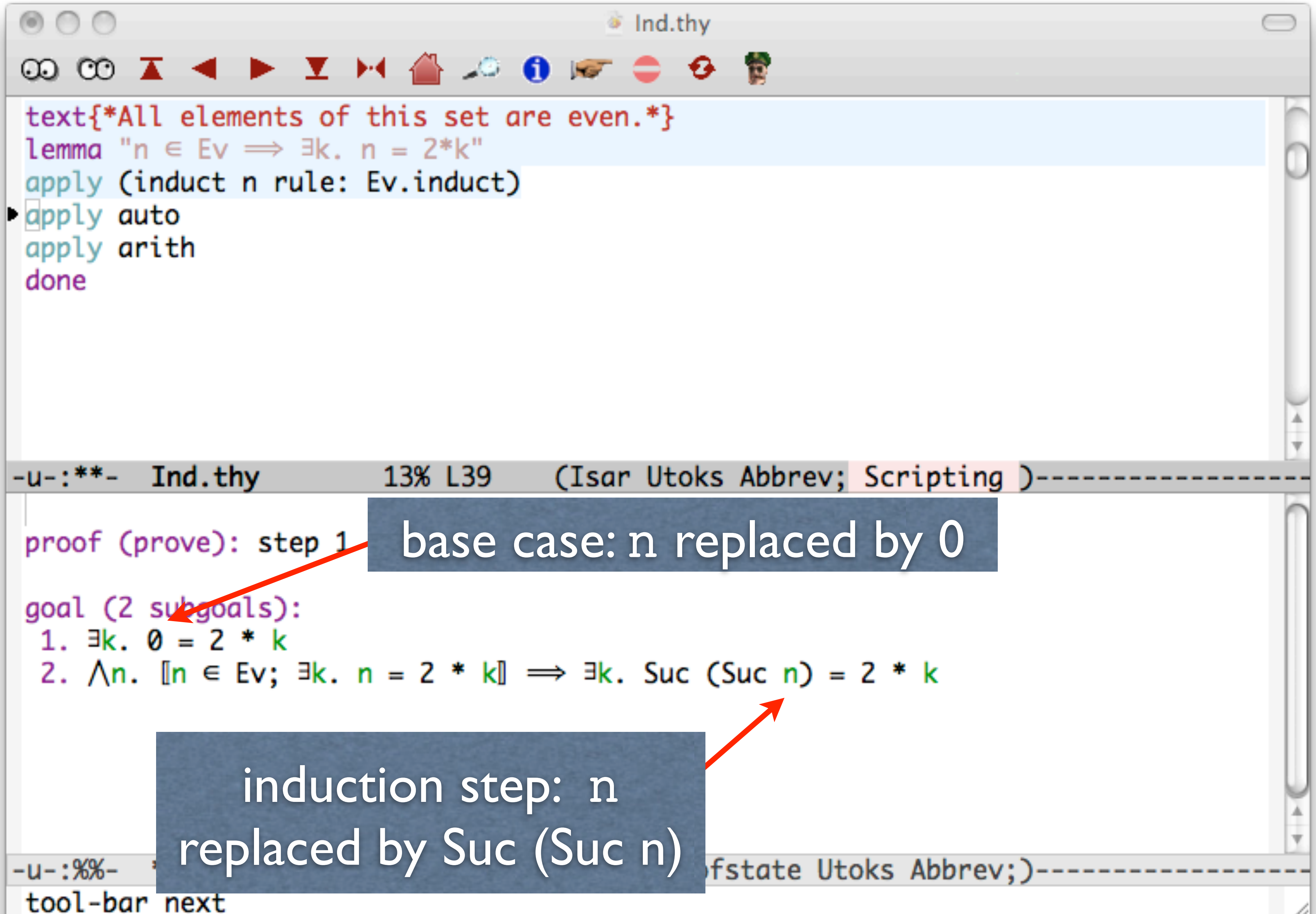
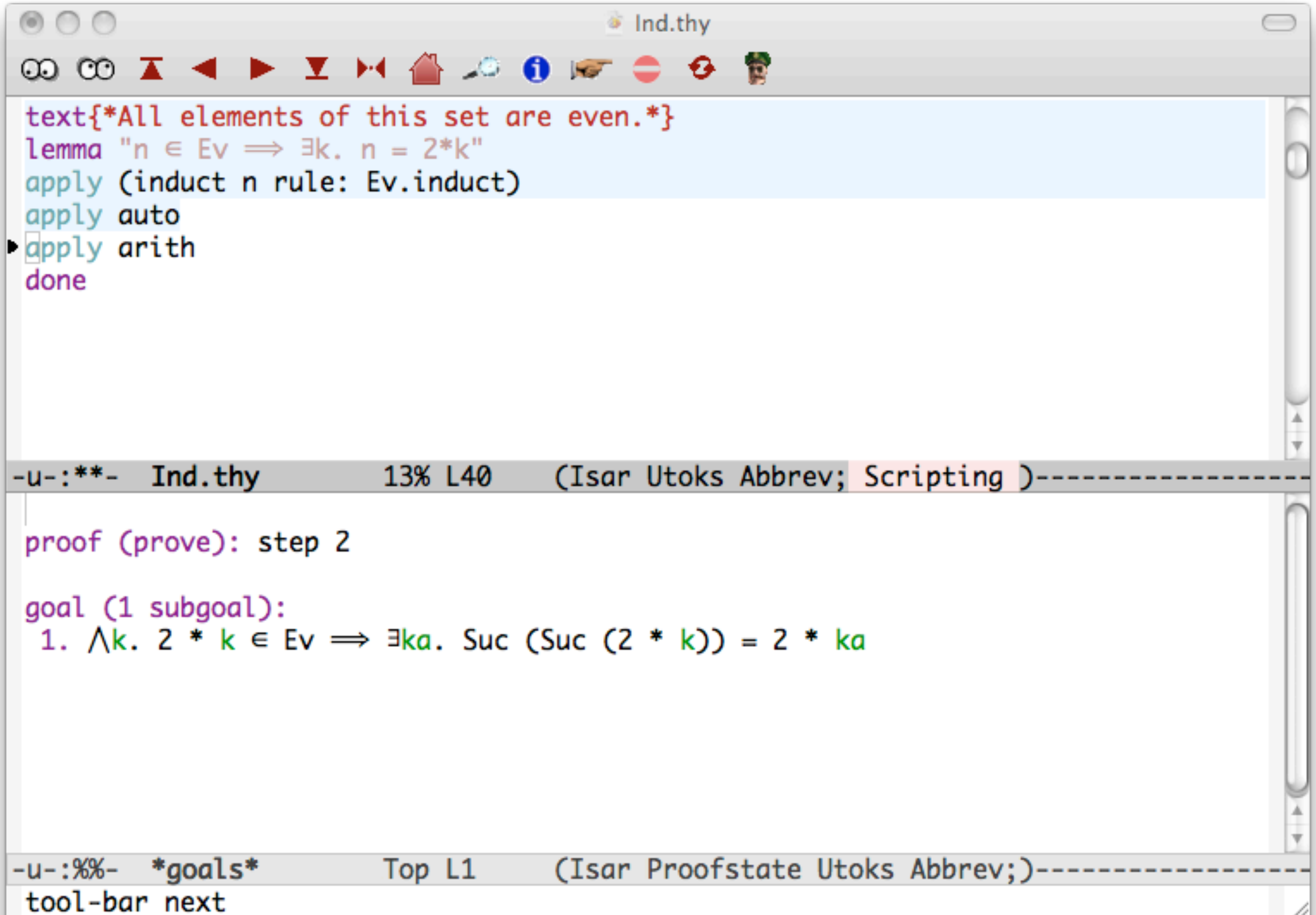


Nearly There!



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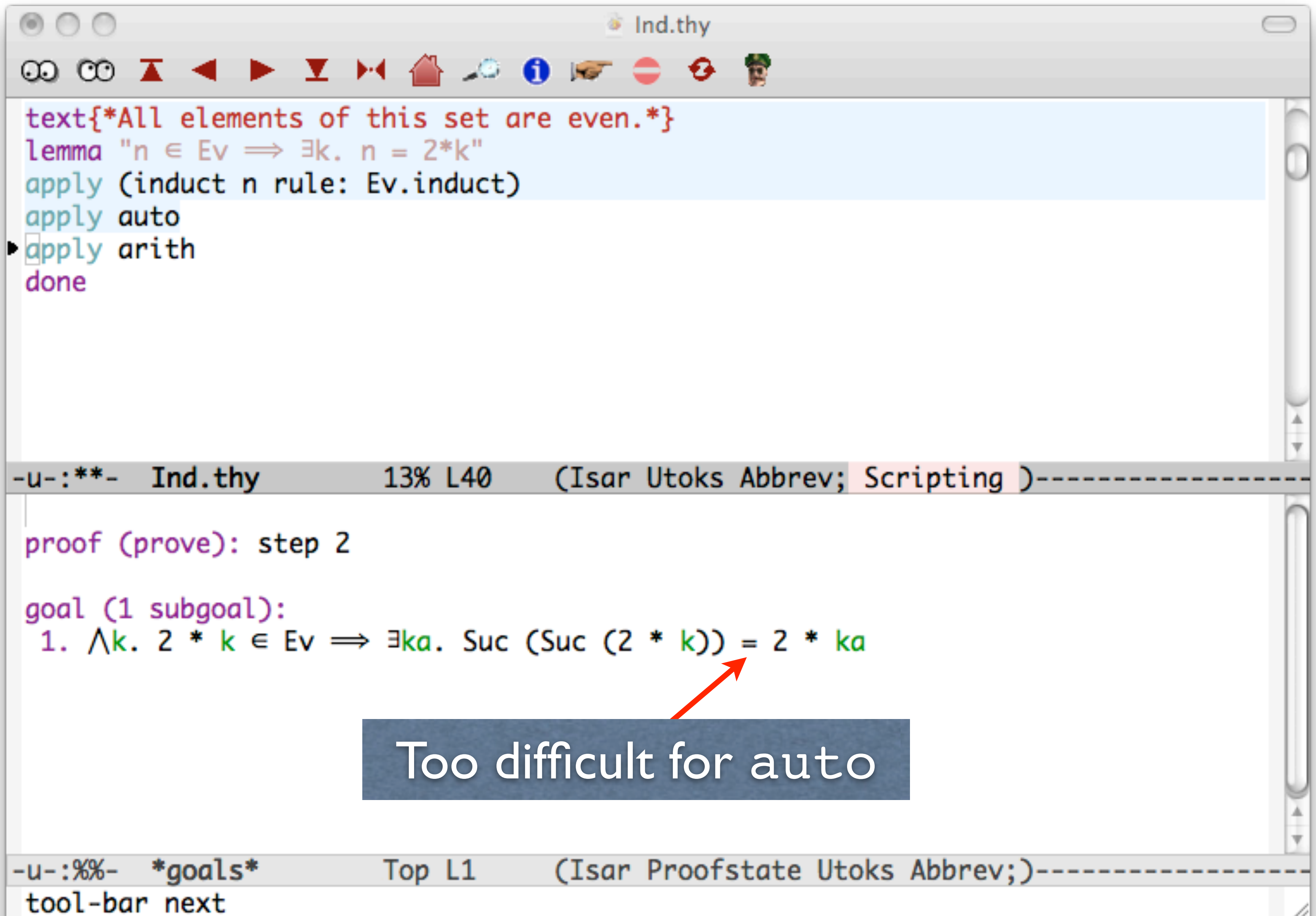

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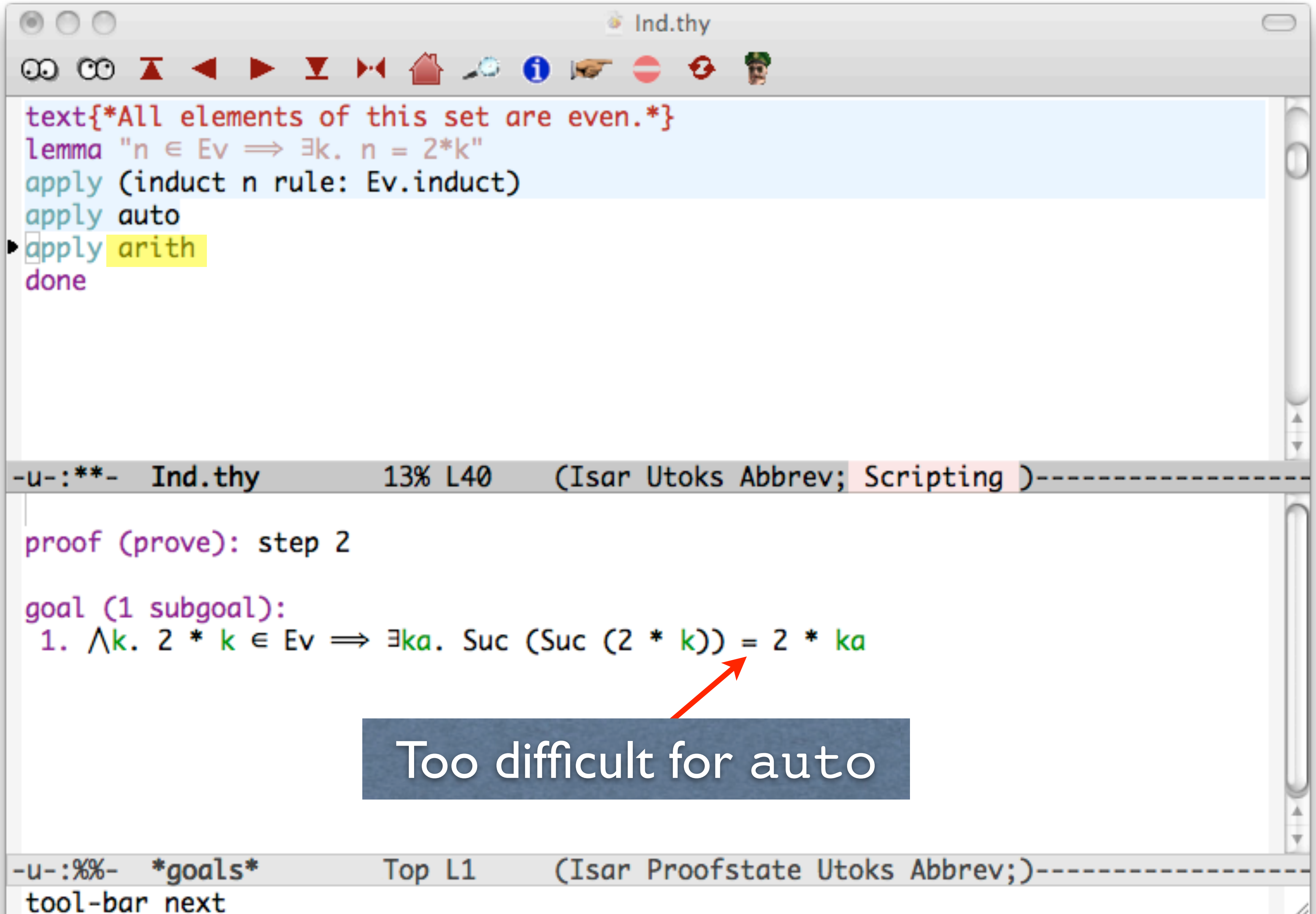
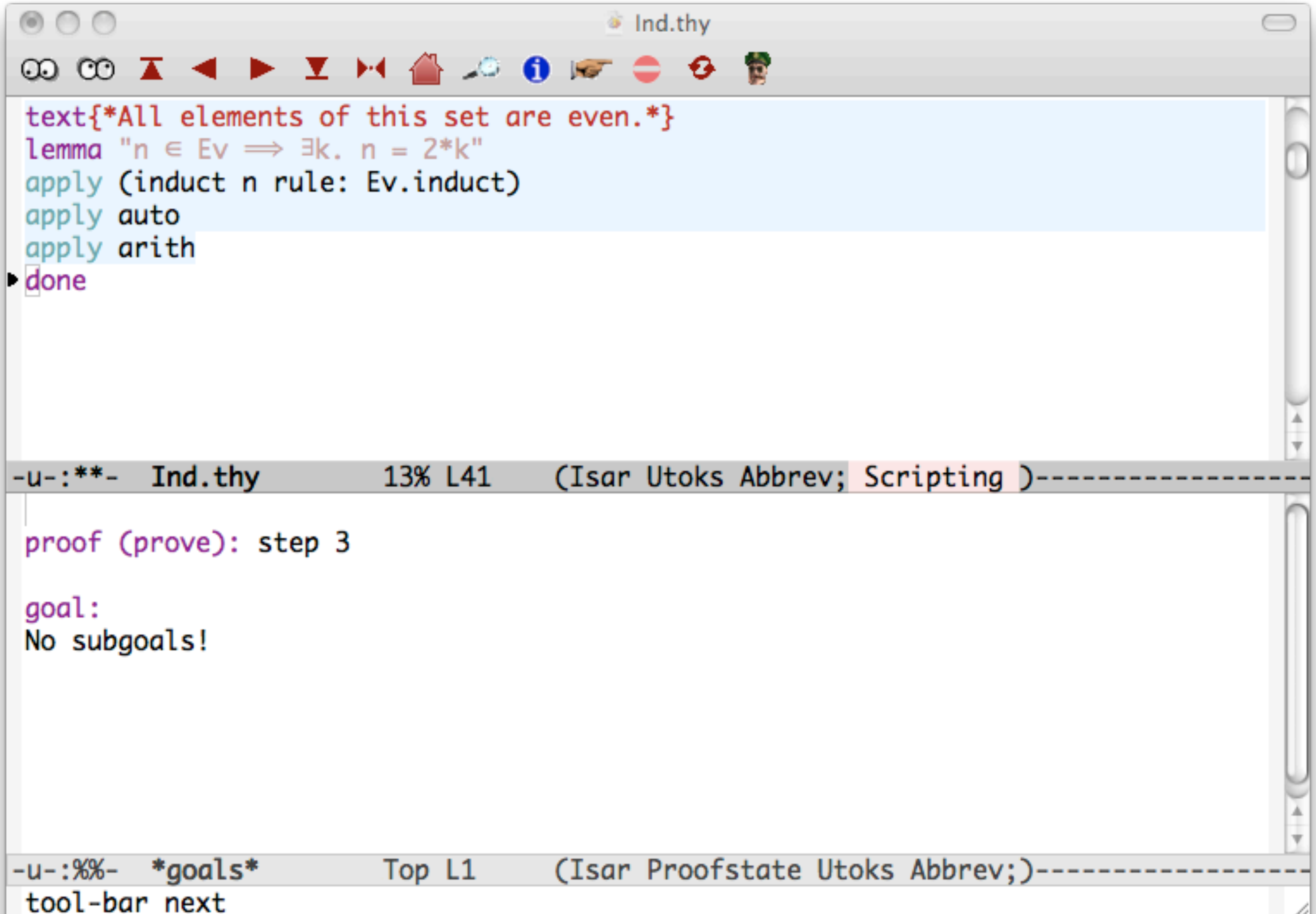


The `arith` Proof Method



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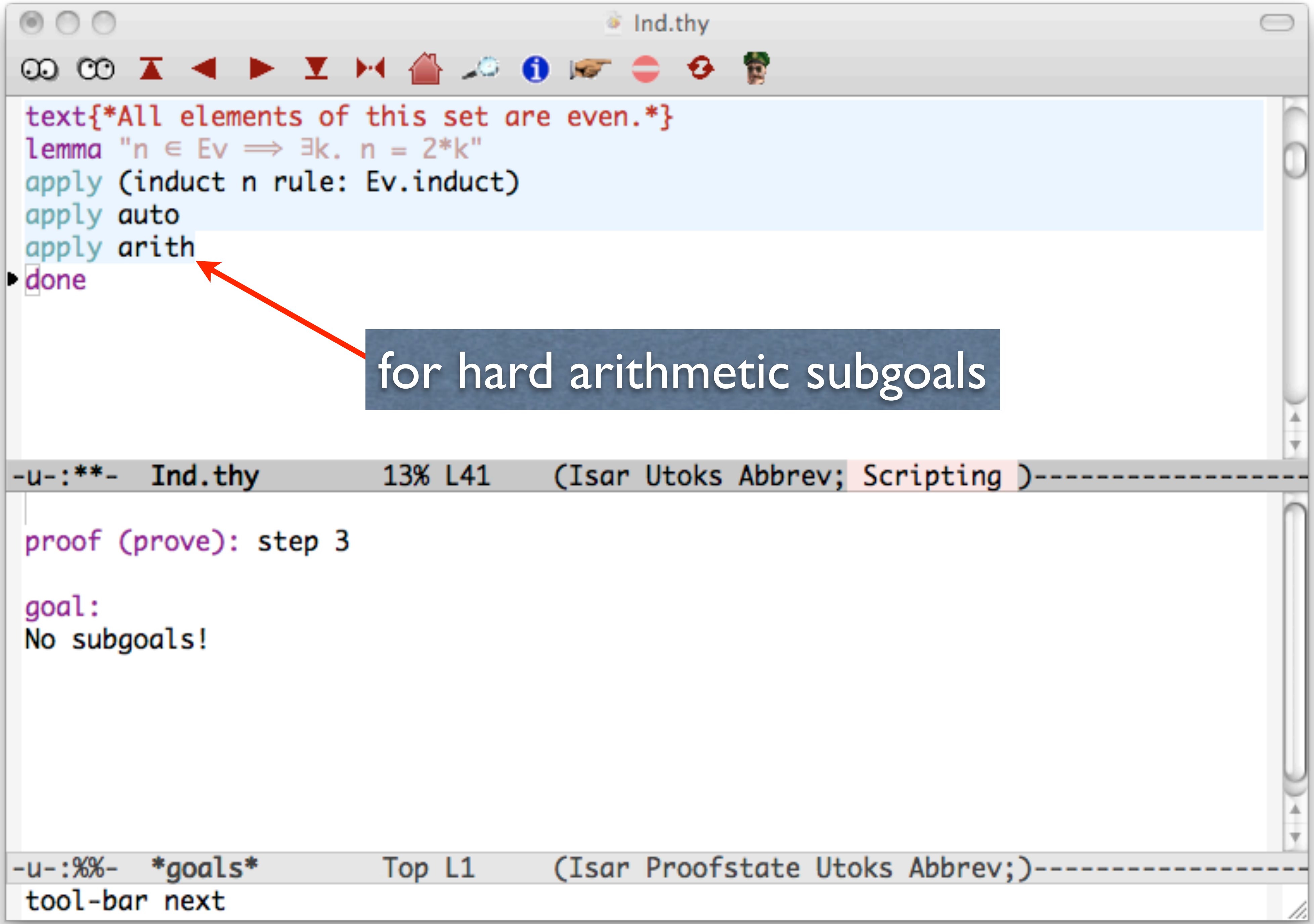
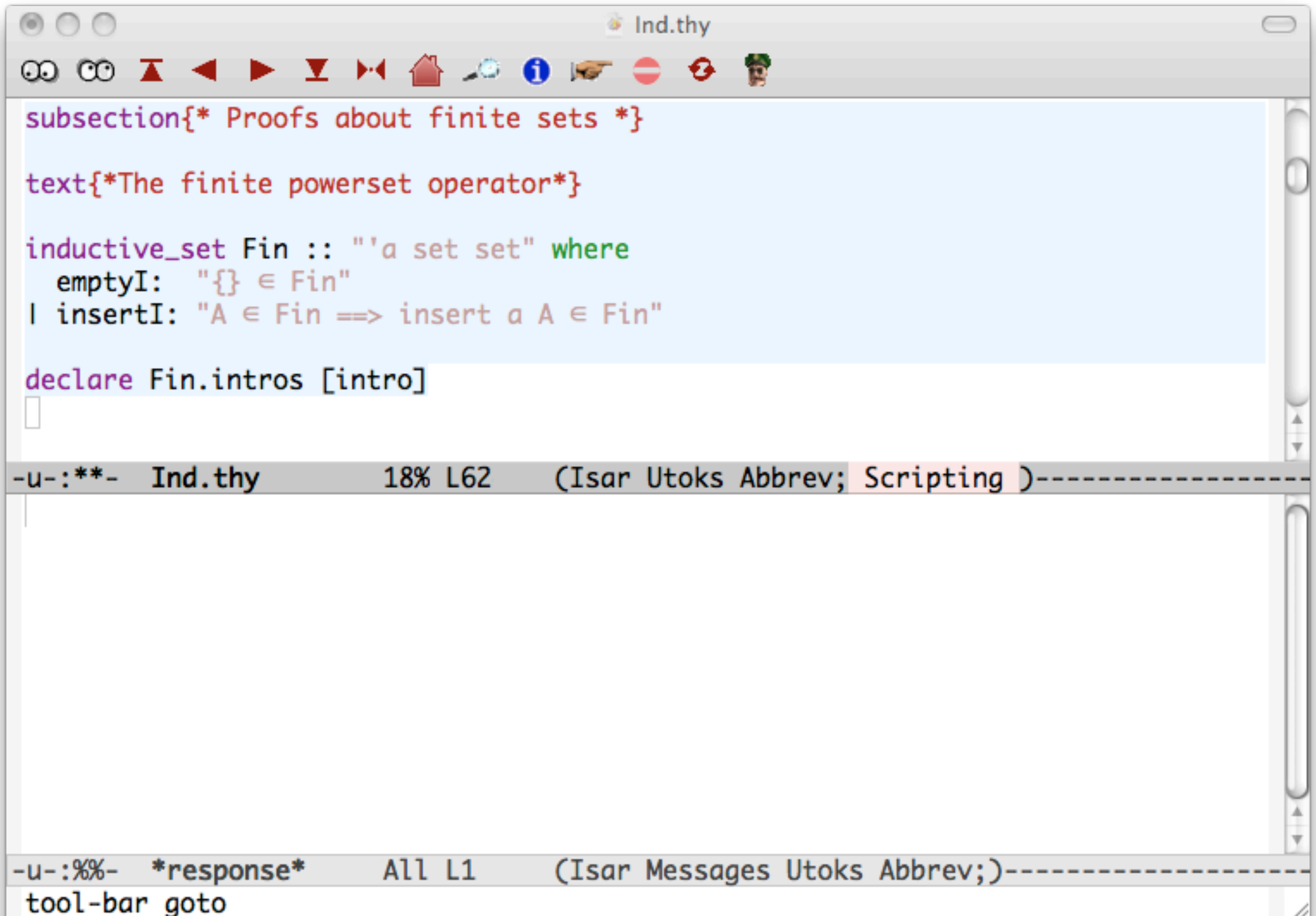


Defining Finiteness



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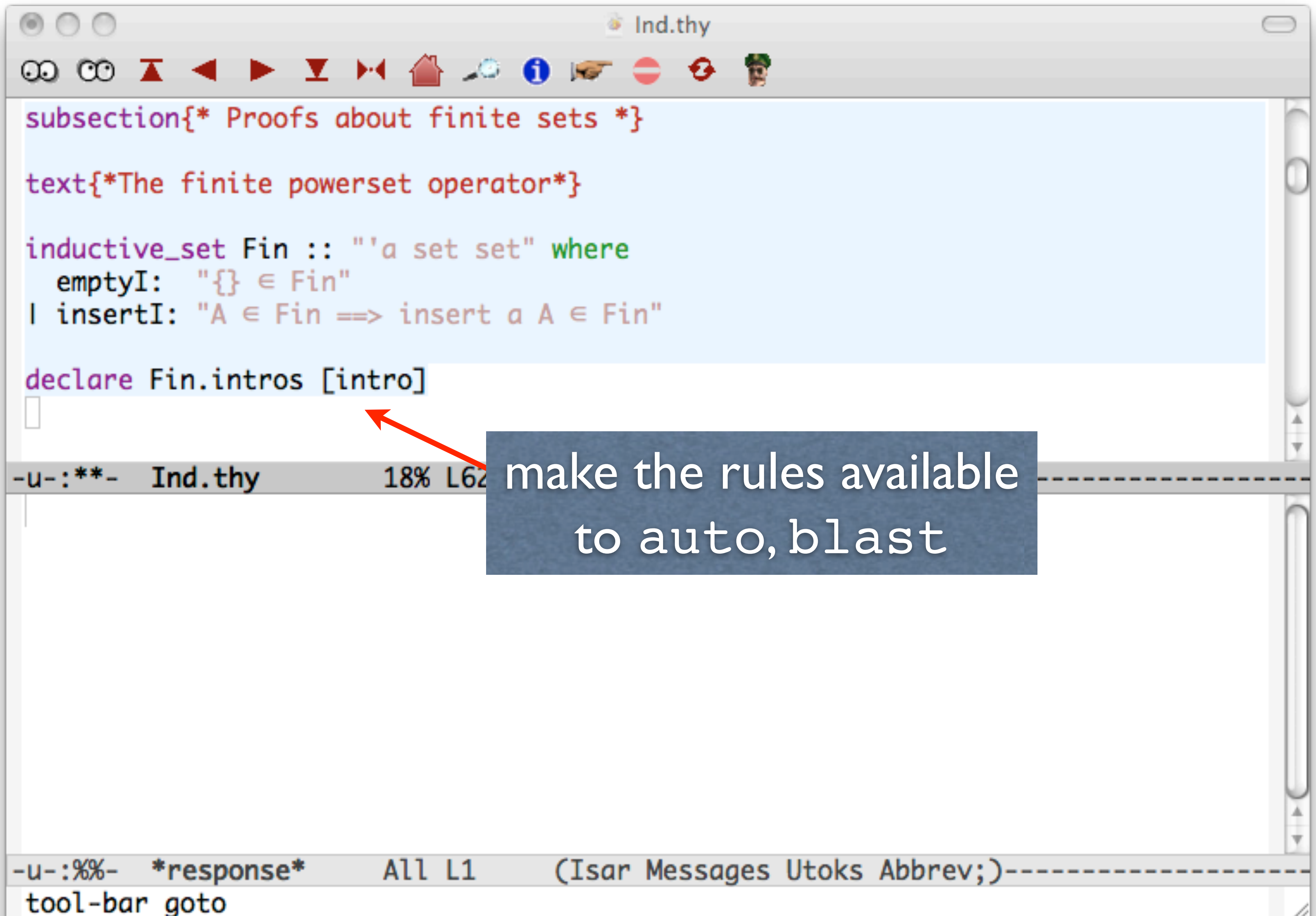
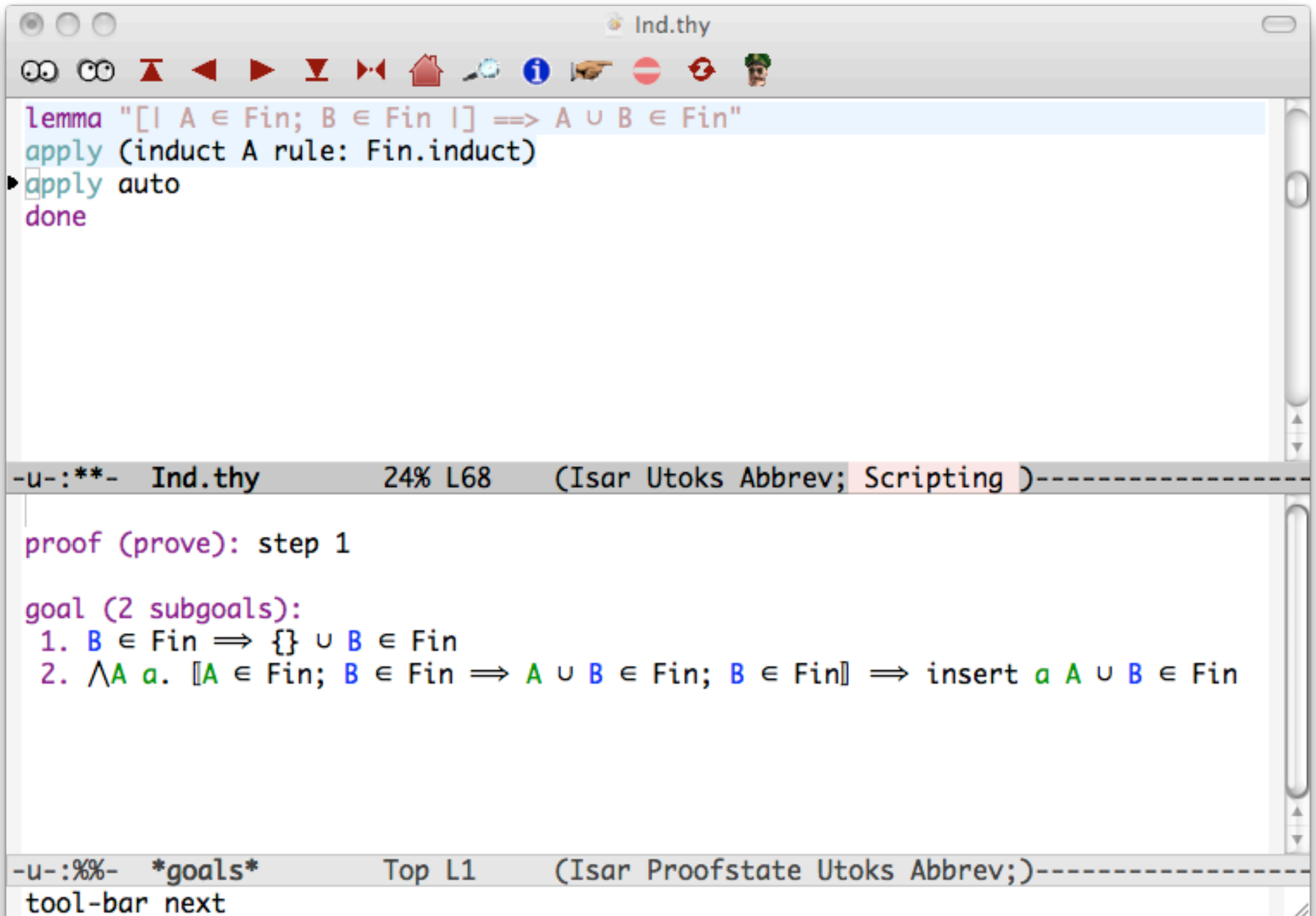


The Union of Two Finite Sets



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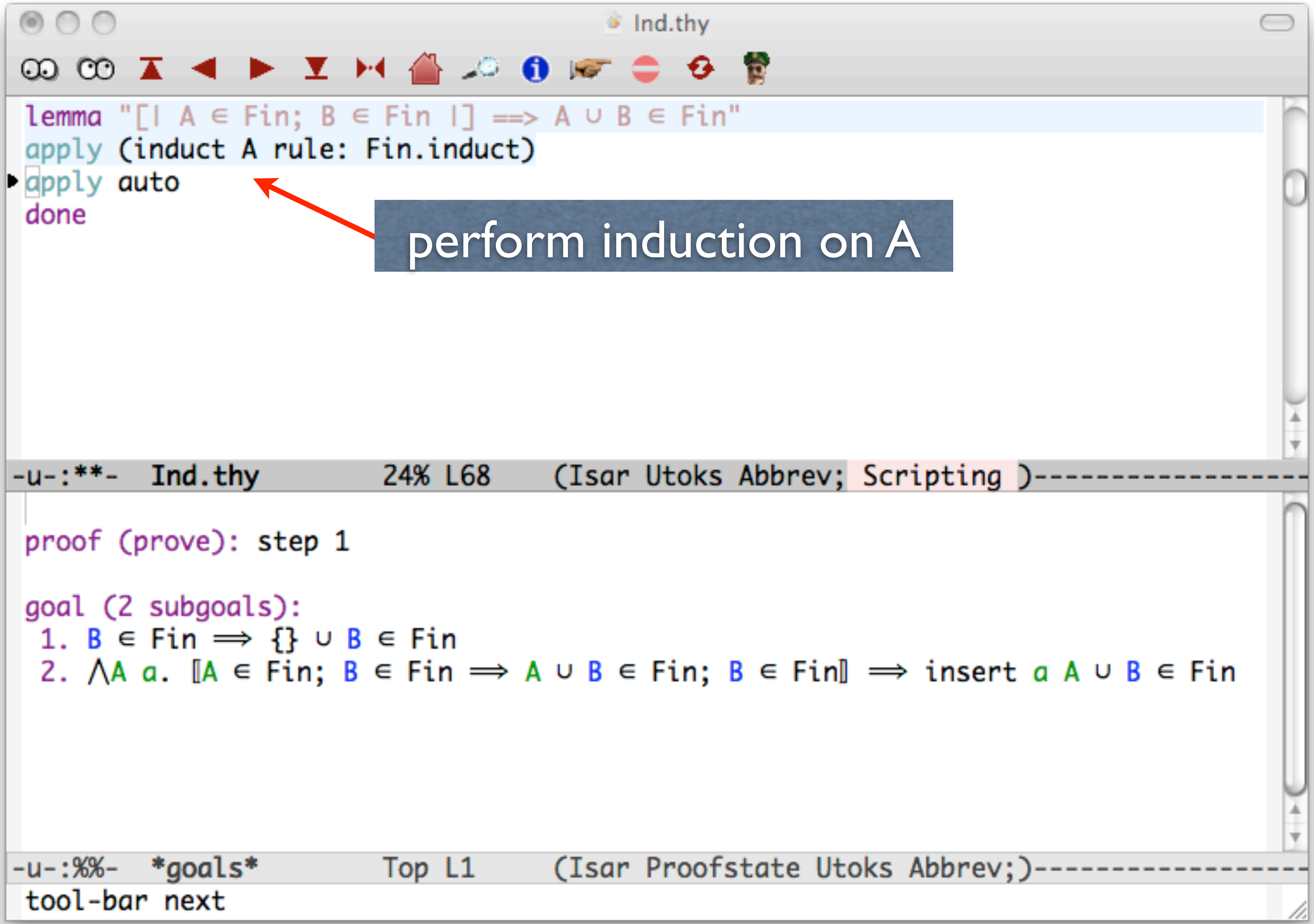

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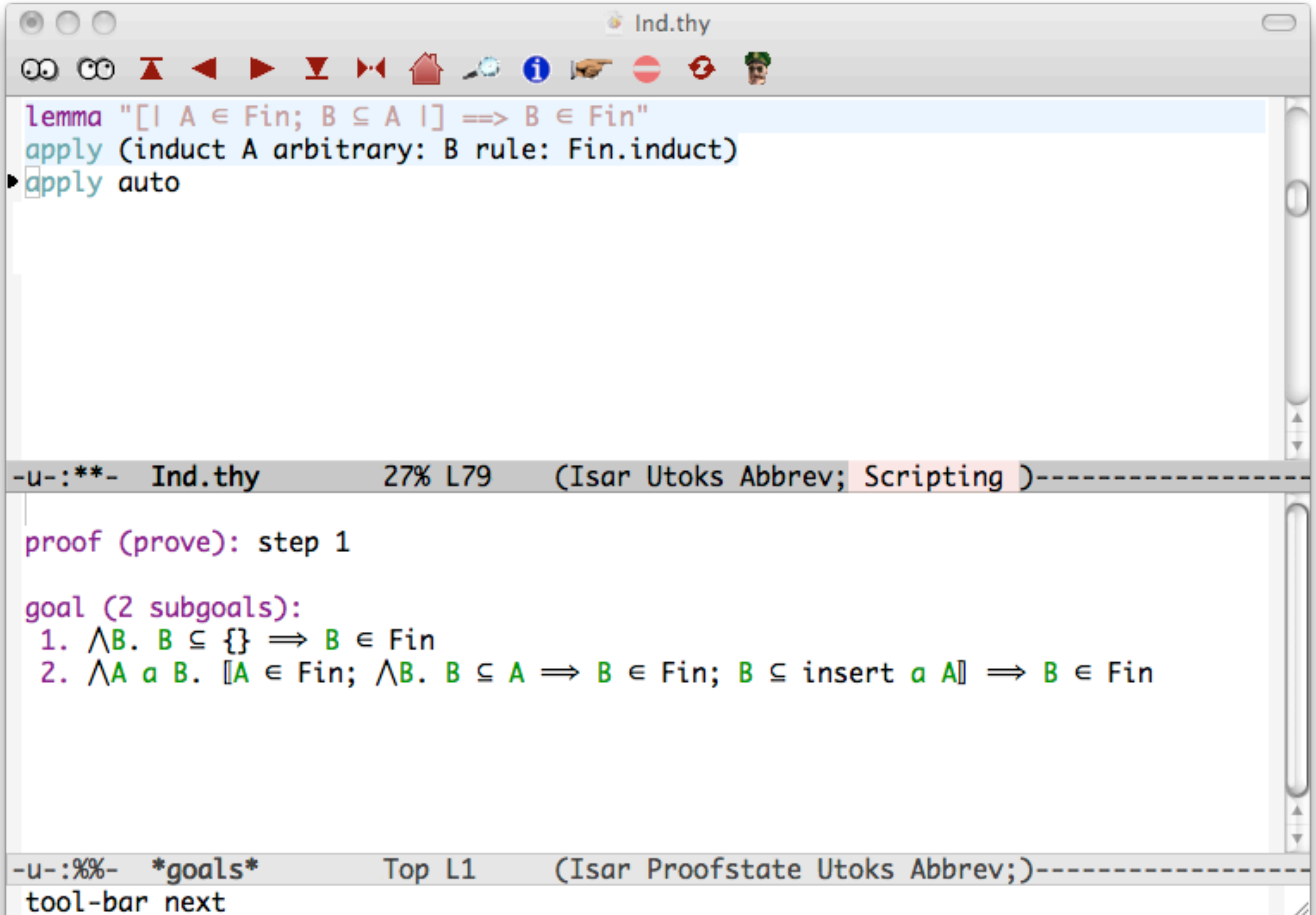

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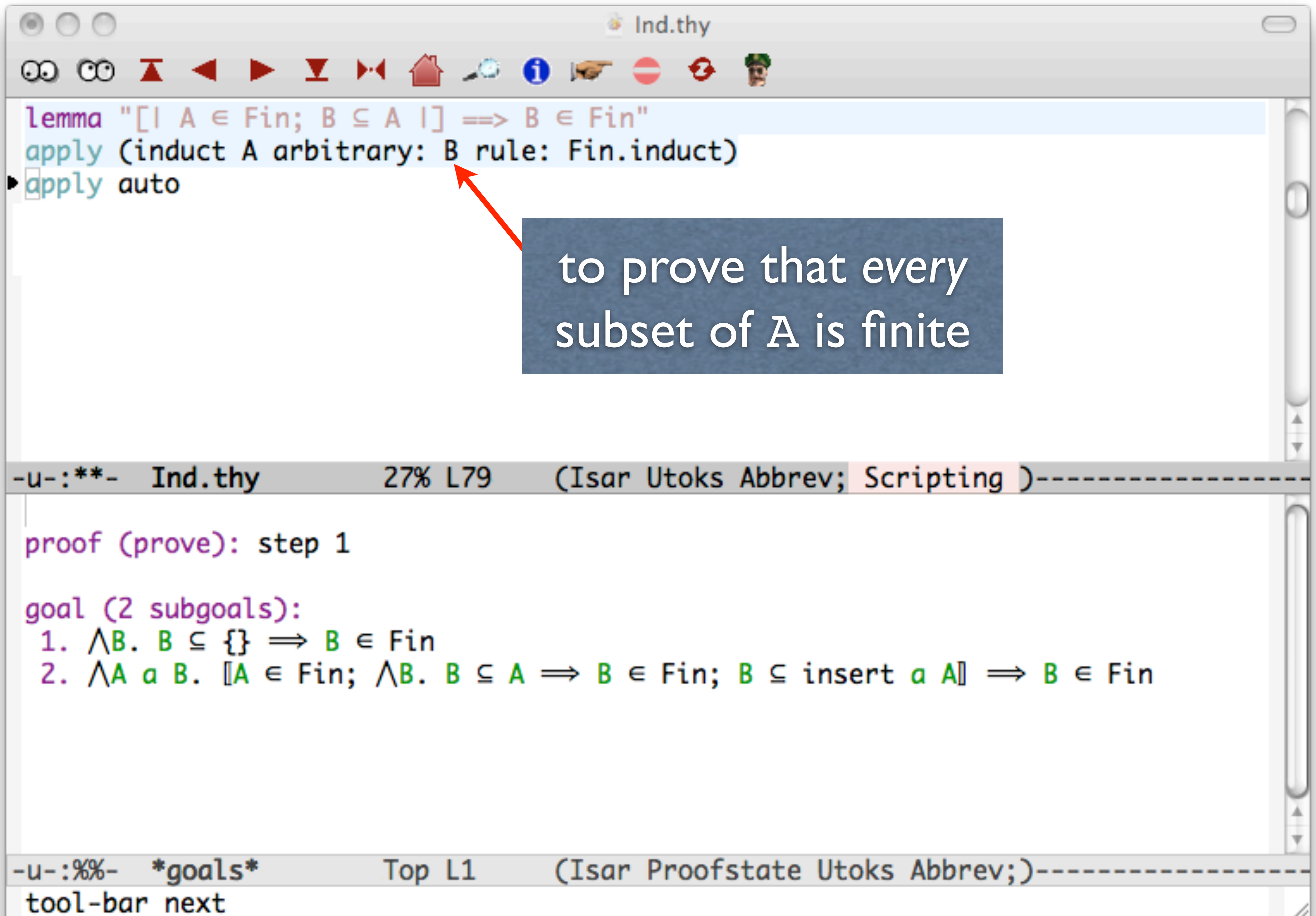

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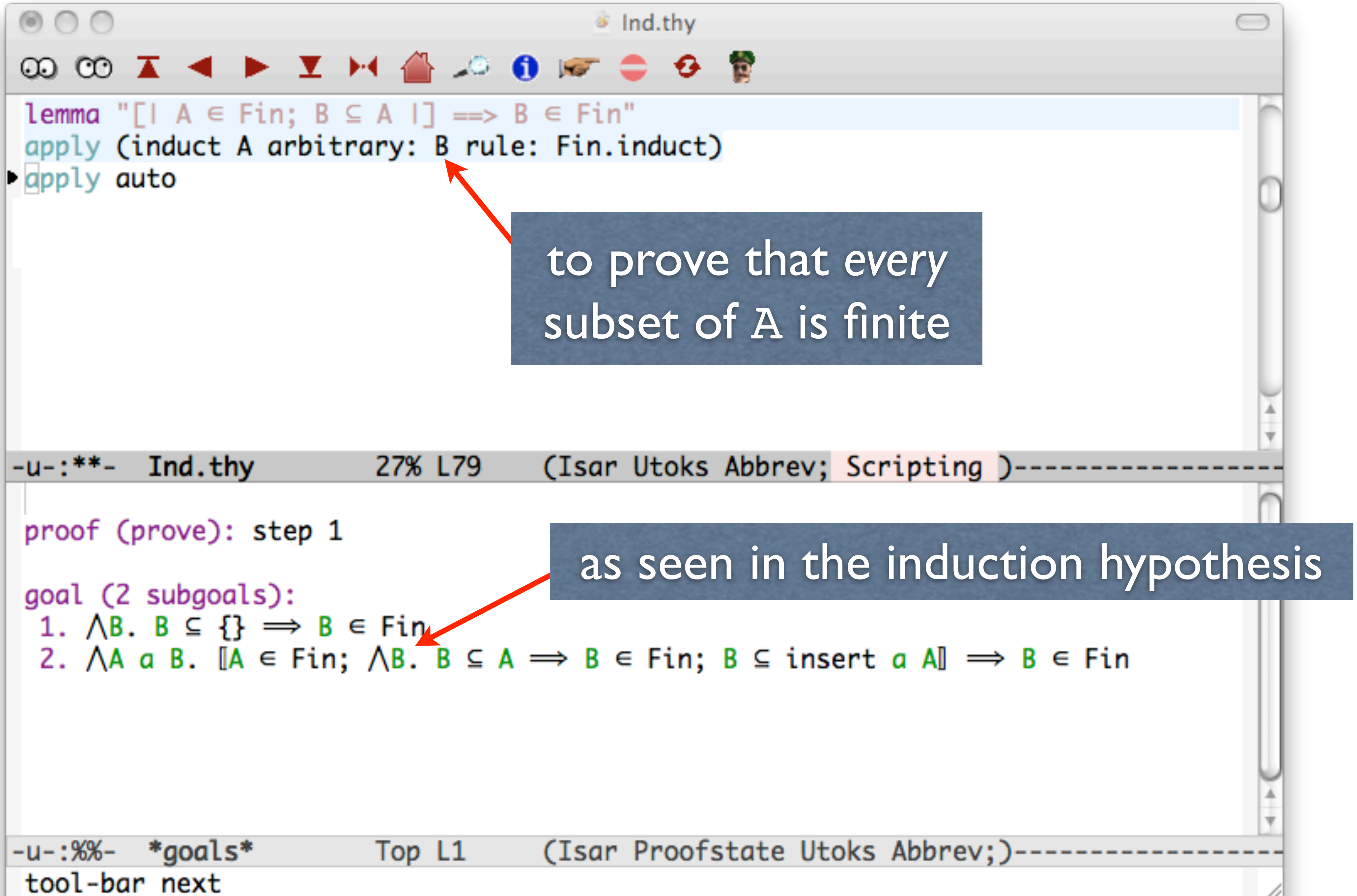
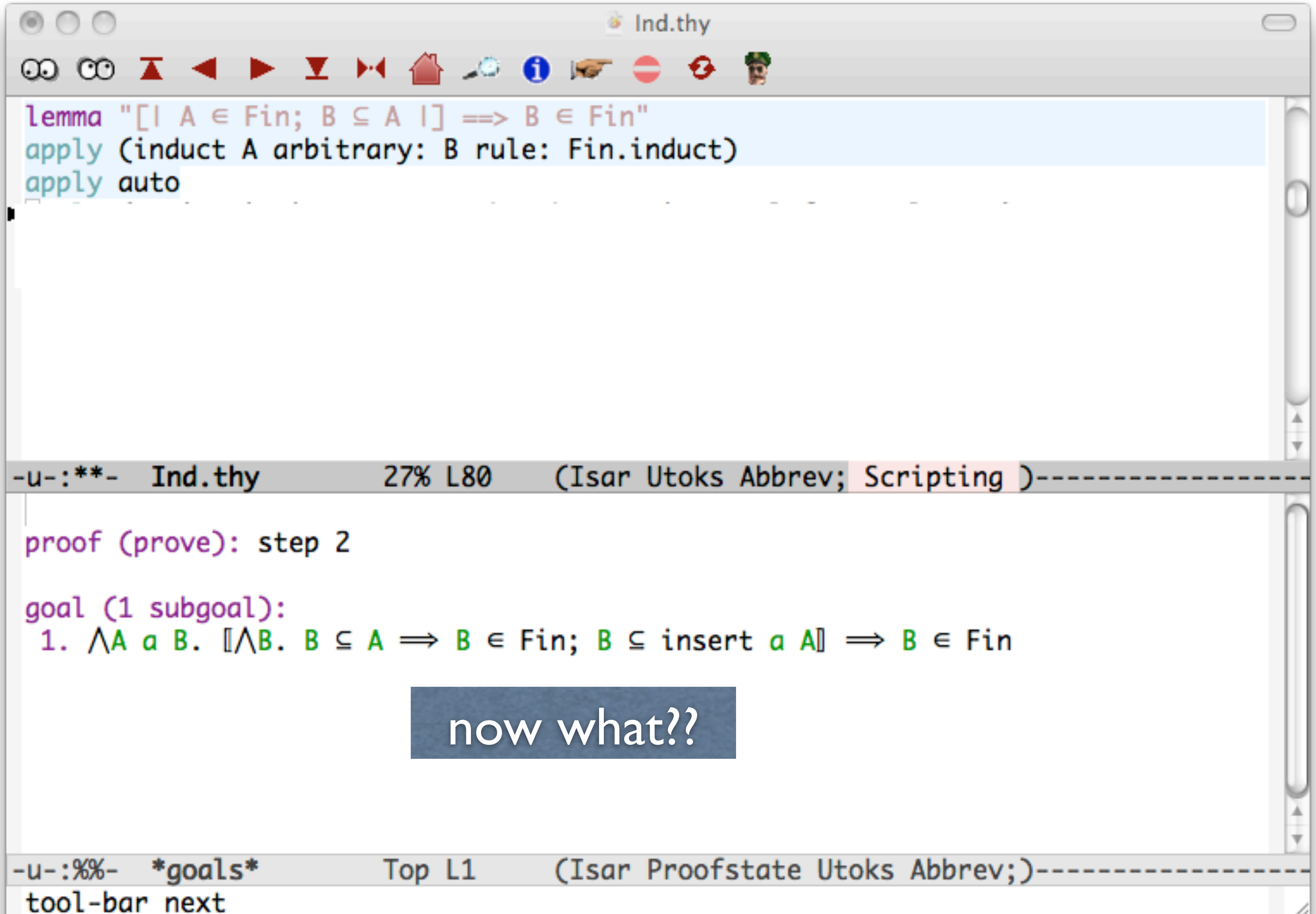
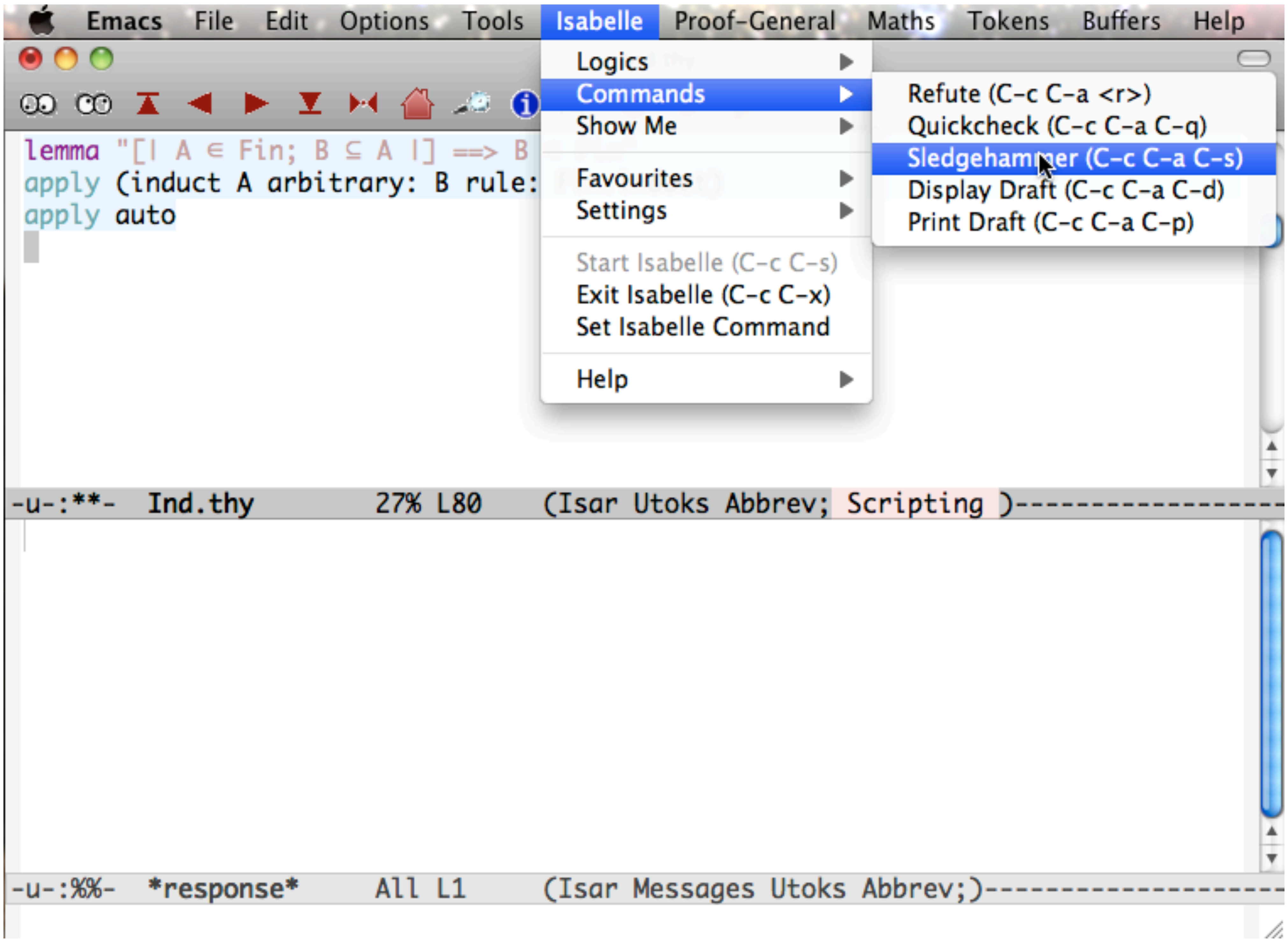


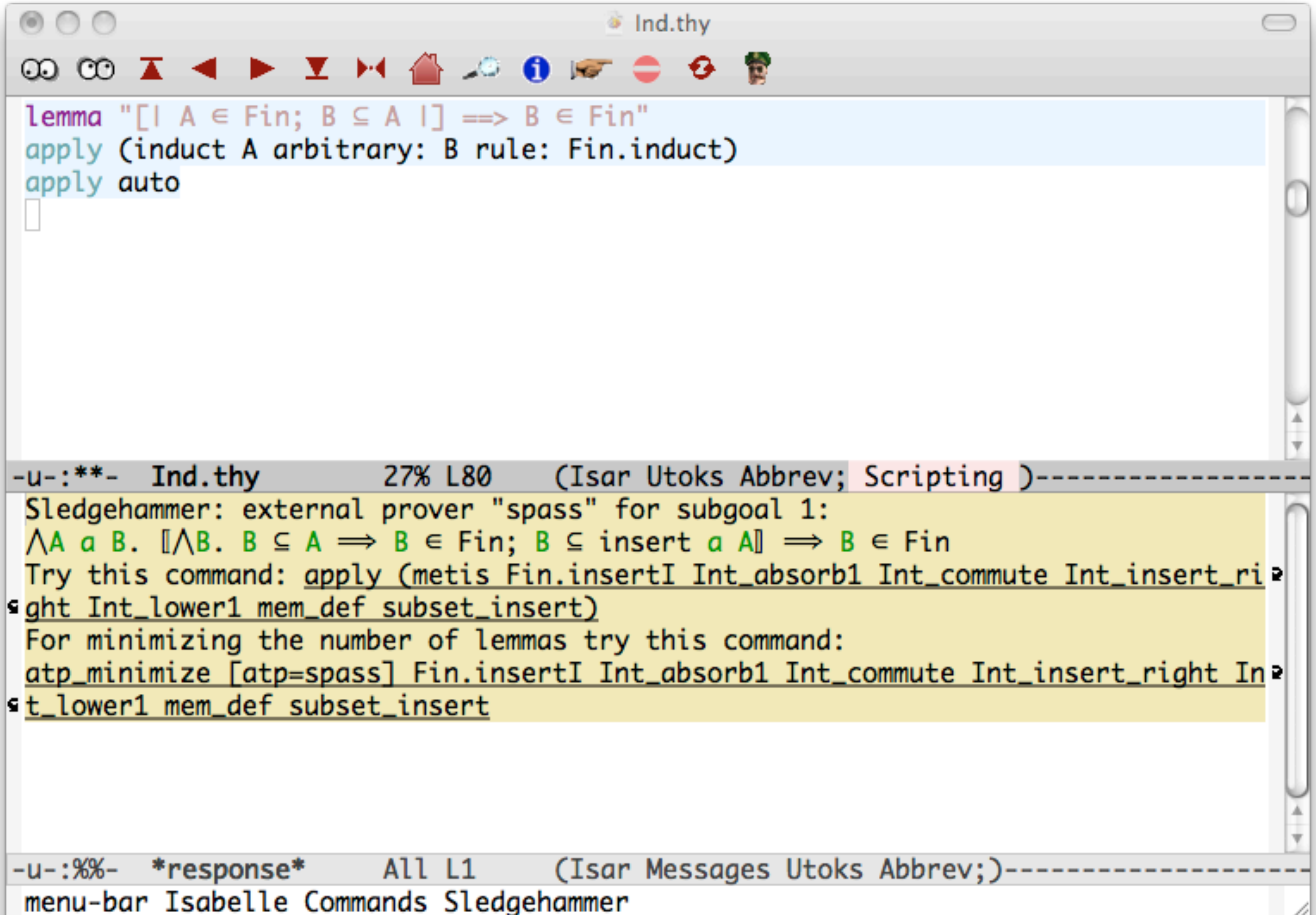
A Crucial Point in the Proof



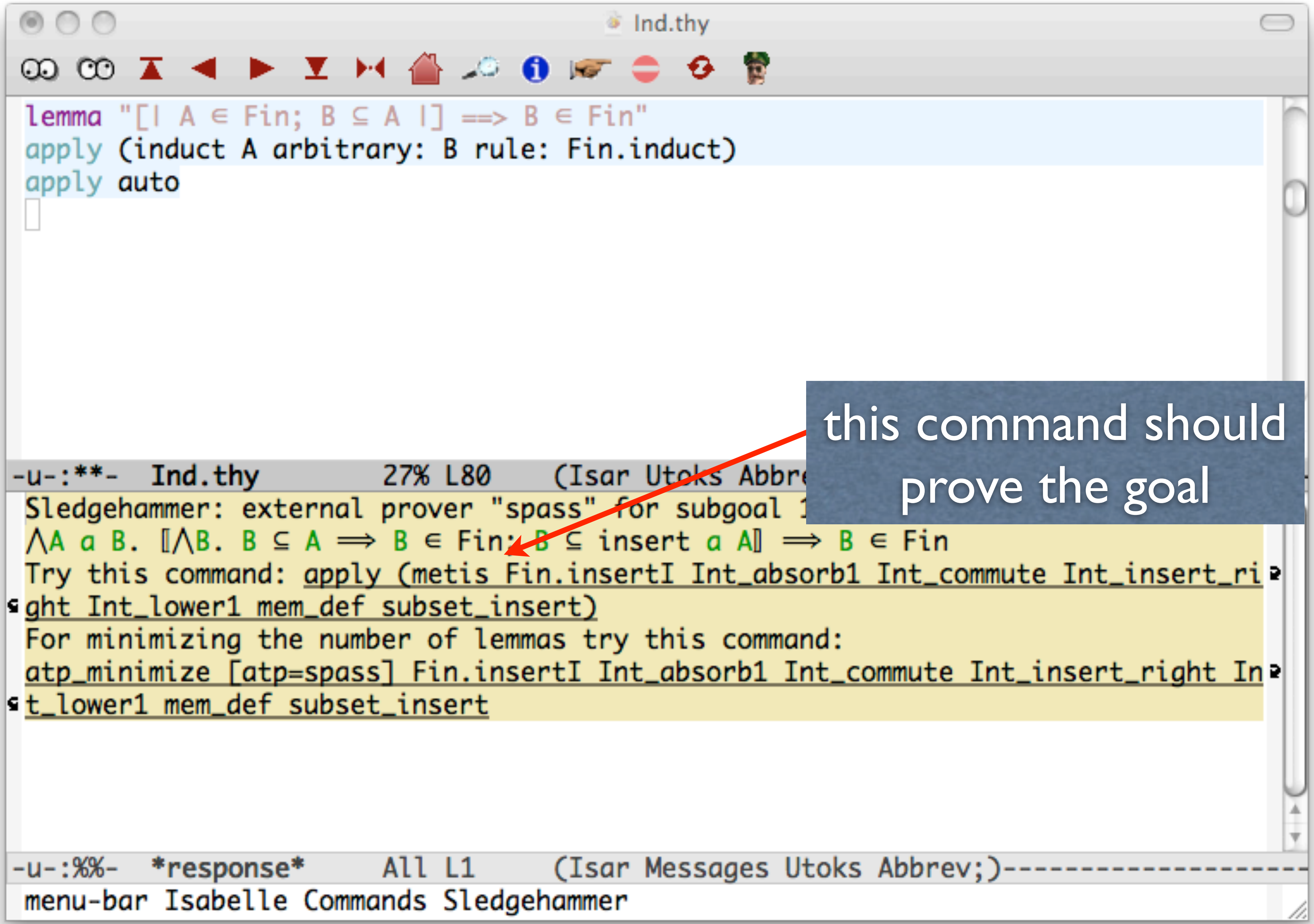
Time to Try Sledgehammer!



Success!


Ind.thy
lemma "[| A ∈ Fin; B ⊆ A |] ==> B ∈ Fin"
apply (induct A arbitrary: B rule: Fin.induct)
apply auto
-u-:**- Ind.thy 27% L80 (Isar Utoks Abbrev; Scripting)
Sledgehammer: external prover "spass" for subgoal 1:
⋀A a B. ⟦⋀B. B ⊆ A ⟹ B ∈ Fin; B ⊆ insert a A⟧ ⟹ B ∈ Fin
Try this command: apply (metis Fin.insertI Int_absorb1 Int_commute Int_insert_right Int_lower1 mem_def subset_insert)
For minimizing the number of lemmas try this command:
atp_minimize [atp=spass] Fin.insertI Int_absorb1 Int_commute Int_insert_right Int_lower1 mem_def subset_insert
-u-:%%- *response* All L1 (Isar Messages Utoks Abbrev;)
menu-bar Isabelle Commands Sledgehammer

Success!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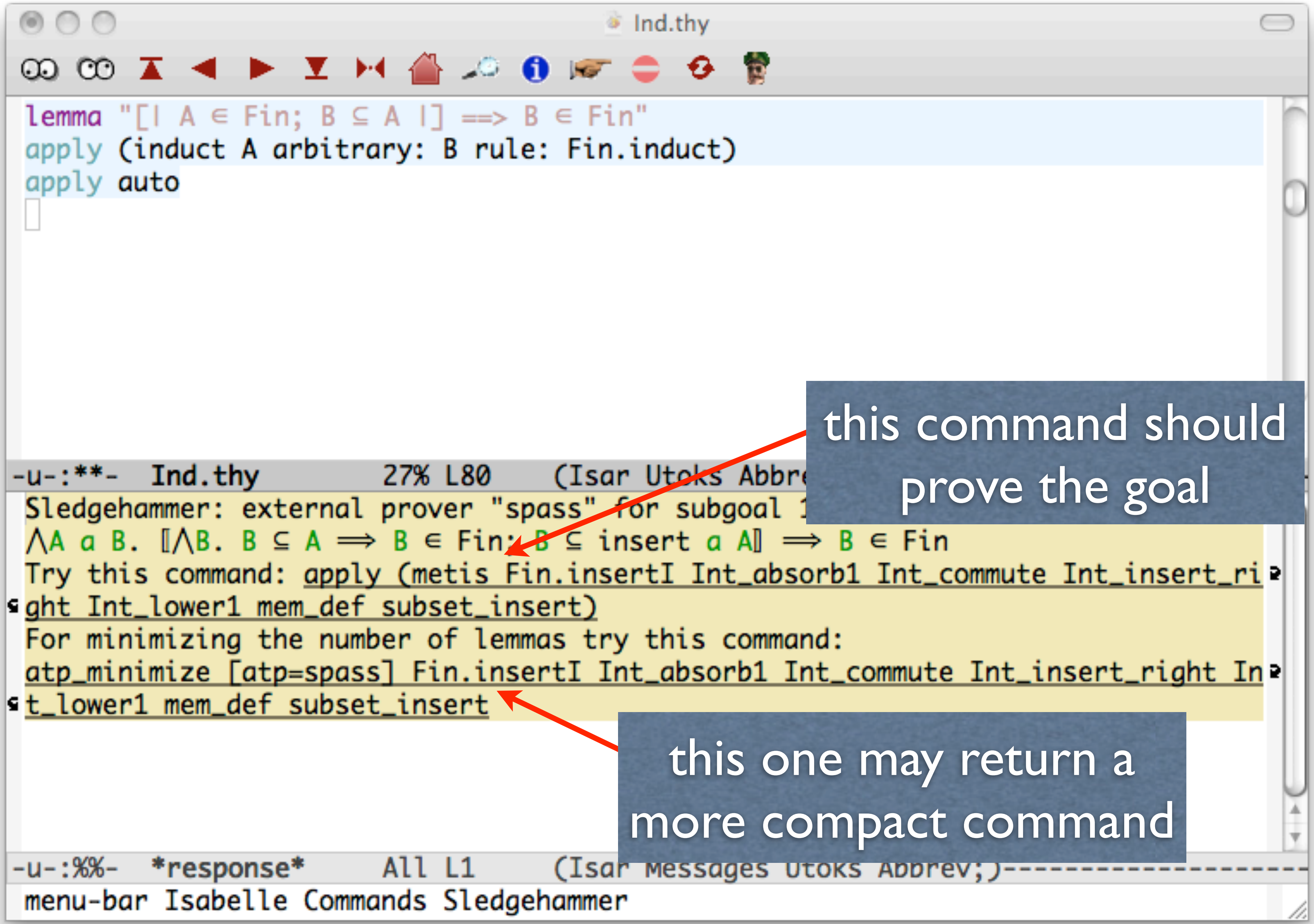

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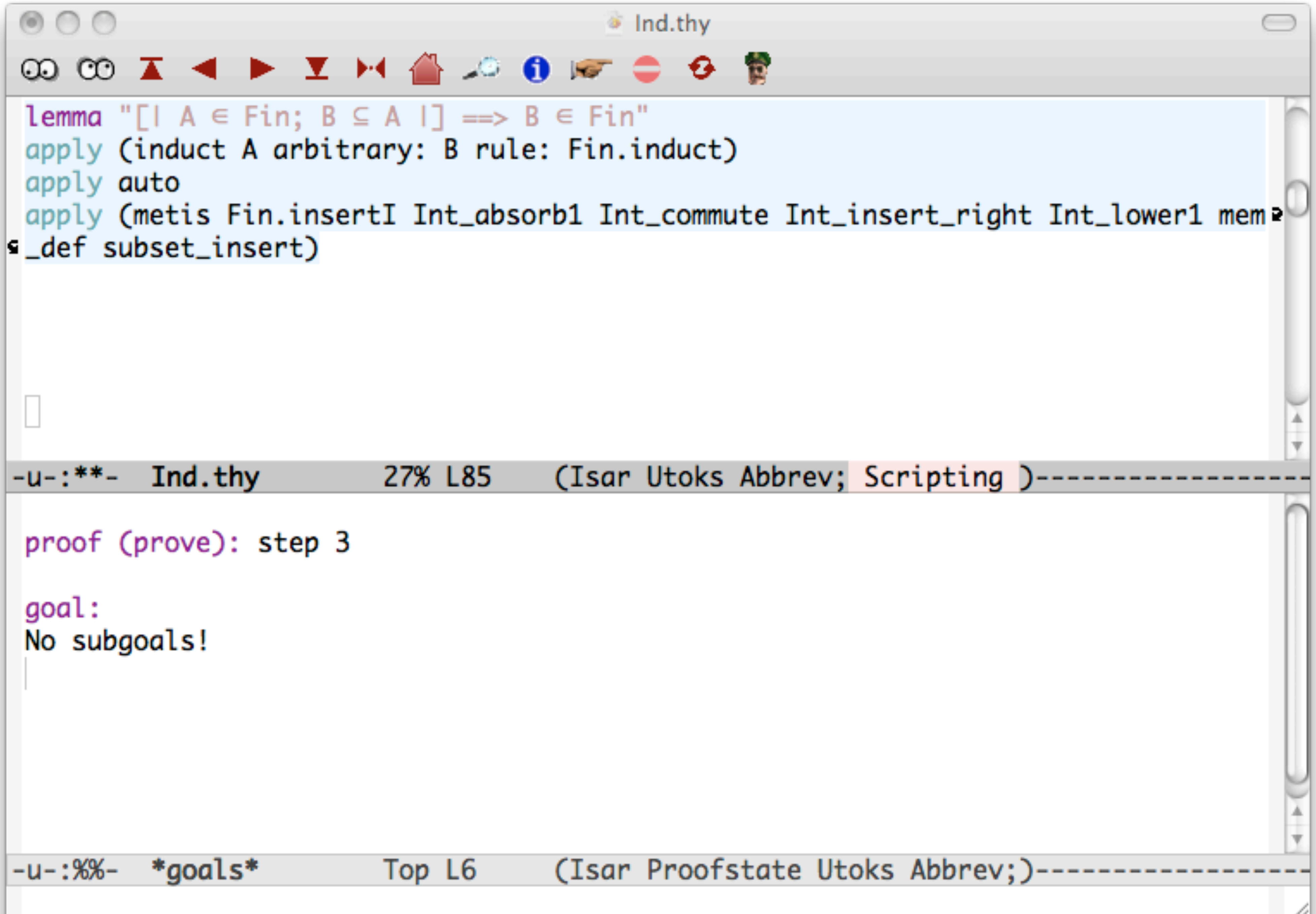
menu-bar Isabelle Commands Sledgehammer
this command should prove the goal
this one may return a more compact command

The Completed Proof



How Sledgehammer Works

Isabelle

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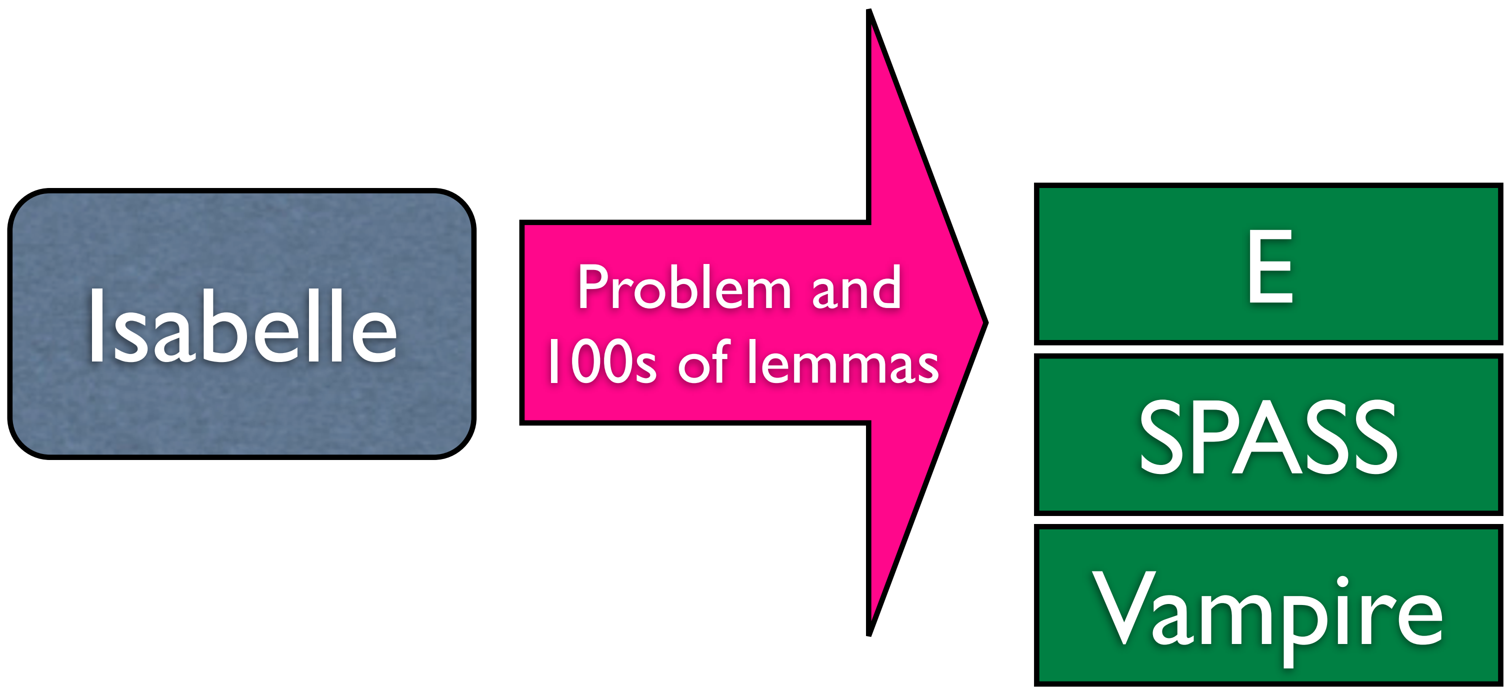

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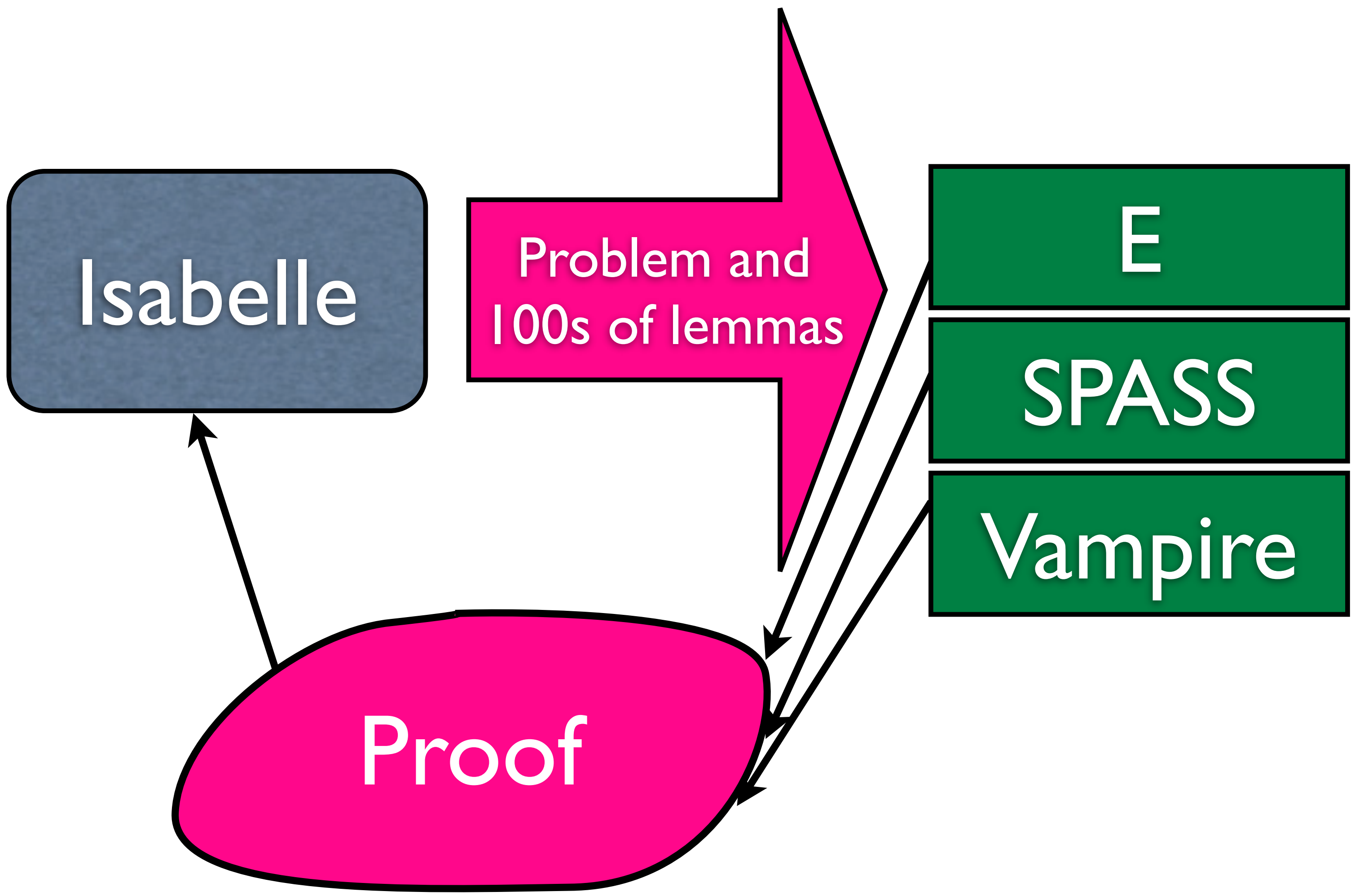

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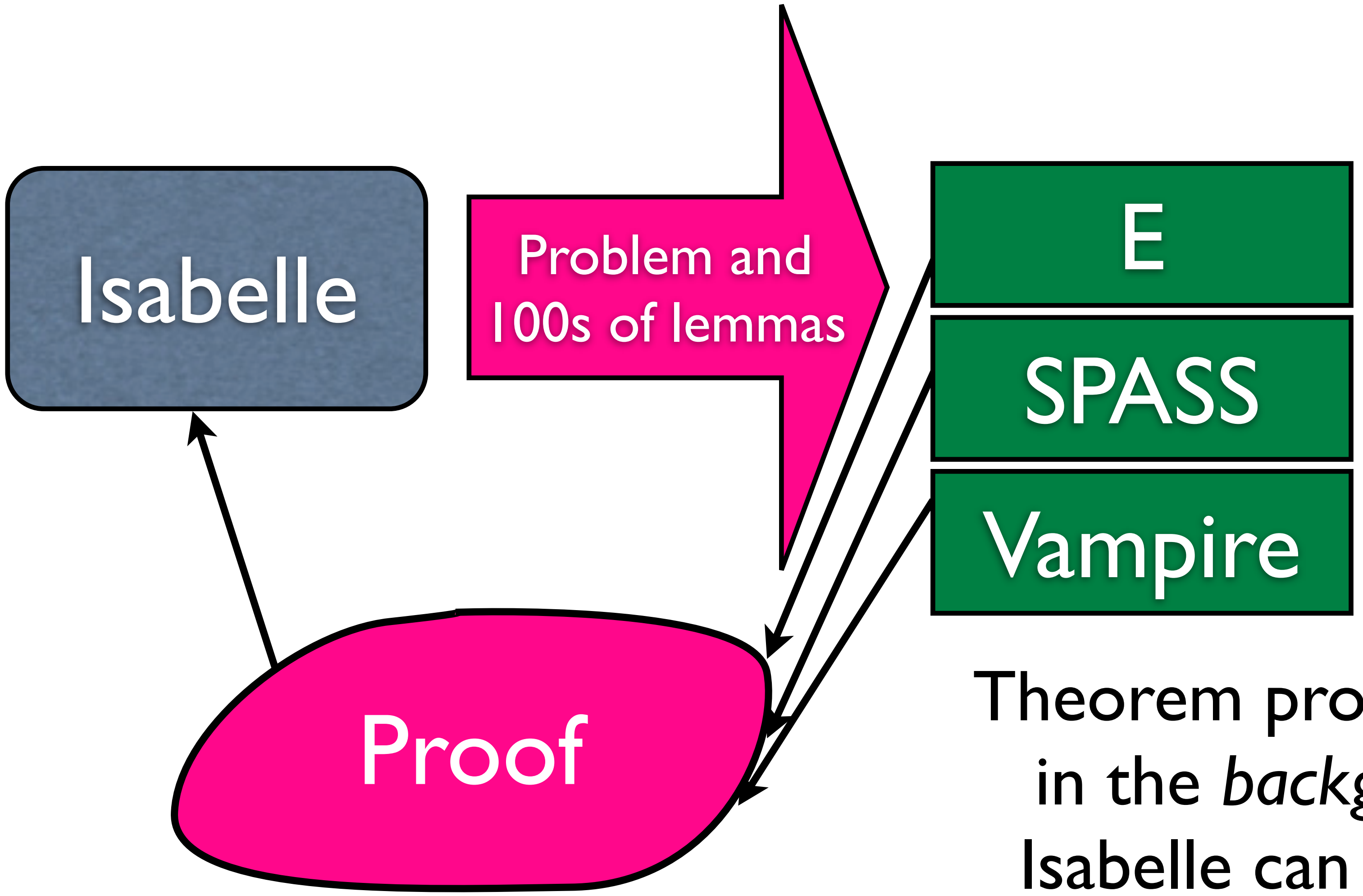

Theorem provers run in the *background*. Isabelle can still be used!

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- It does not prove the goal, but returns a call to `metis`. This command *usually* works...
- The minimise option removes redundant theorems, increasing the likelihood of success.
- Calling `metis` directly is difficult unless you know exactly which lemmas are needed.